Ph 444   Solutions for Problem Set 6

1. (Ryden 9.3) This problem examines the recombination of helium in the early universe. For simplicity, it considers a universe containing only He and assumes that the amount of doubly ionized helium is negligible. The latter assumption is reasonable for temperatures close to that at which He becomes neutral. Then the relevant reaction is $He + \gamma \leftrightarrow He^+ + e^-$. The ionization energy for He is $Q_{He} = 24.6$ eV. Applying the Maxwell-Boltzmann equation for the number densities of He, He^+, and e^- yields, in analogy with Ryden equation (9.22)

$$\frac{n_{He}}{n_{He^+}n_e} = \frac{g_{He}}{g_{He^+}g_e}\left(\frac{m_{He}}{m_{He^+}m_e}\right)^{3/2}\left(\frac{kT}{2\pi\hbar^2}\right)^{-3/2}\exp\left(\frac{[m_{He^+}+m_e-m_{He}]c^2}{kT}\right). \quad (1)$$

Now $m_{He} \simeq m_{He^+}$, the ratio of statistical weights is 1/4, and the difference in the rest masses is Q_{He}. Thus,

$$\frac{n_{He}}{n_{He^+}n_e} = \frac{1}{4}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2}\exp\left(\frac{Q_{He}}{kT}\right). \quad (2)$$

This is the equivalent of Ryden equation (9.23) for He. The fractional ionization of He is defined as

$$X \equiv \frac{n_{He^+}}{n_{He^+}+n_{He}} = \frac{4n_{He^+}}{n_{baryon}} = \frac{4n_e}{n_{baryon}}. \quad (3)$$

The above equation uses the recommended $n_{baryon} = 4(n_{He^+} + n_{He})$. Equation 3 implies

$$n_{He} = \frac{1-X}{X}n_{He^+} = \frac{1-X}{X}n_e \quad (4)$$

and so equation 2 becomes

$$\frac{1-X}{X} = \frac{n_{He^+}}{4}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2}\exp\left(\frac{Q_{He}}{kT}\right). \quad (5)$$

Now equation 3 also implies that photon-to-baryon ratio can be written as

$$\eta \equiv \frac{n_{baryon}}{n_\gamma} = \frac{4n_{He^+}}{Xn_\gamma} = \frac{4n_{He^+}}{0.243X}\left(\frac{\hbar c}{kT}\right)^3. \quad (6)$$

The last step in the above equation uses the formula for the number density of blackbody photons at temperature T (Ryden equation 9.27). Using equation 6 to eliminate n_{He^+} from equation 5 produces the final equation for X in terms of η and T:

$$\frac{1-X}{X} = \frac{0.243X\eta}{4\times 4}\left(\frac{kT}{\hbar c}\right)^3\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2}\exp\left(\frac{Q_{He}}{kT}\right) \quad (7)$$

$$\Rightarrow \frac{1-X}{X^2} = 0.239\eta\left(\frac{kT}{m_e c^2}\right)^{3/2}\exp\left(\frac{Q_{He}}{kT}\right). \quad (8)$$

This equation must be solved numerically for the T that corresponds to a given X. It is useful to introduce the dimensionless variable $y \equiv Q_{He}/(kT)$ and take the logarithm of both sides of equation 8:

$$\ln\left(\frac{1-X}{X^2}\right) = \ln(0.239\eta) + \frac{3}{2}\ln\left(\frac{Q_{He}}{m_e c^2}\right) - \frac{3}{2}\ln(y) + y \qquad (9)$$

$$\Rightarrow \ln\left(\frac{1-X}{X^2}\right) = -16.34 + \ln(\eta) - \frac{3}{2}\ln(y) + y \qquad (10)$$

For $X = 1/2$ and $\eta = 5.5 \times 10^{-10}$, this becomes

$$38.36 = y - \frac{3}{2}\ln(y). \qquad (11)$$

The solution is $y = 44.04$. One way to solve this is to write the equation as $y = 33.36 + (3/2)\ln(y)$. Plugging a guess for y into the right-hand side, calculating a new value of y, and then repeating the process quickly converges to the solution. For example, $y = 38.36, 43.83, 44.03, 44.04, 44.04, \ldots$ This usually works if the right-hand-side is a slowly-varying function of y. It is easy to see why this procedure converges to the right answer by plotting the two sides of the equation and then graphically sketching what the iteration procedure is doing. Thus, the temperature for $X = 0.5$ is $kT = Q_{He}/44.04 = 0.559$ eV $\Rightarrow T = 6.48 \times 10^3$ K. This temperature is about $1.7\times$ higher than that at which hydrogen is recombining.

2. (Ryden 9.5) If the universe is ionized between the time t_* and today, t_0, then the optical depth to electron scattering produced by a uniform density of baryons is

$$\tau = \int_{t_*}^{t_0} n_e(t)\sigma_e c dt. \qquad (12)$$

Here n_e is the electron density, σ_e is the Thompson scattering cross section, 6.65×10^{-29} m^2, and c is the speed of light. If the ionization is complete and we assume that universe contains only hydrogen, then the number density of electrons equals the number density of baryons, $n_e = n_b$. Now $n_{b,0} = \Omega_{b,0}\rho_{crit}/m_p$. Calculating the critical density for $H_0 = 70$ km s Mpc^{-1}, adopting $\Omega_{b,0} = 0.04$, and plugging in the proton mass yields $n_{b,0} = 0.22$ m^{-3}, which is the value used by Ryden in chapter 9. Finally, $n_b(t) = n_{b,0}(a_0/a)^3$ and for a flat, matter-dominated universe $a/a_0 = (t/t_0)^{2/3}$. Putting all of this together,

$$\tau = \int_{t_*}^{t_0} n_{b,0}(t_0/t)^2\sigma_e c dt \qquad (13)$$

$$= n_{b,0}\sigma_e c t_0 \int_{t_*/t_0}^{1} \frac{dt'}{(t')^2} \qquad (14)$$

$$= n_{b,0}\sigma_e c t_0 \left(\frac{t_0}{t_*} - 1\right). \qquad (15)$$

The value of t_* corresponding to a given value of τ is

$$t_* = t_0\left(\frac{\tau}{n_{b,0}\sigma_e c t_0} + 1\right)^{-1}. \qquad (16)$$

The value of t_0 that is consistent with the assumption of a matter-dominated universe is $t_0 = 2/(3H_0) = (2/3)(4.4\times 10^{17}\text{ s}) = 2.9\times 10^{17}\text{ s} = 9.3\times 10^9$ yrs. Plugging in the numerical values for $\tau = 1$ yields $t_*/t_0 = 1.3\times 10^{-3}$ and $t_* = 1.2\times 10^7$ yrs. This corresponds to the redshift

$$\begin{aligned} 1+z_* &= \left(\frac{a_0}{a_*}\right) = \left(\frac{t_0}{t_*}\right)^{2/3} && (17)\\ &= \left(\frac{1}{n_{b,0}\sigma_e c t_0}+1\right)^{2/3} && (18)\\ &= \left(\frac{1}{1.3\times 10^{-3}}\right)^{2/3} = 85. && (19)\end{aligned}$$

Thus, $z_* = 84$.

Some of the class adopted the actual age of the universe, $t_0 = 13.6$ Gyrs, which yields $t_* = 2.6\times 10^7$ yrs and $z_* = 65$. A few intrepid souls even used Ryden equation (6.29) relating $a(t)$ and t for a universe containing matter and a cosmological constant to calculate z from t_*, which probably is more consistent with a t_0 of 13.6 Gyrs.

Of course, to precisely calculate τ for a flat universe dominated by matter and a cosmological constant, as is ours for the times in question, the correct integral is

$$\begin{aligned} \tau &= \int_{t_*}^{t_0} n_e(t)\sigma_e c dt && (20)\\ &= n_{e,0}\sigma_e c\int_{a_*}^{a_0}\frac{1}{a^3}\frac{da}{da/dt} && (21)\\ &= n_{e,0}\sigma_e c\int_{a_*}^{a_0}\frac{da}{a^3(aH_0[\Omega_{m,0}/a^3+(1-\Omega_{m,0})]^{1/2})} && (22)\\ &= \frac{n_{e,0}\sigma_e c}{H_0}\int_{a_*}^{a_0}\frac{da}{a^{5/2}[\Omega_{m,0}+(1-\Omega_{m,0})a^3]^{1/2}} && (23)\\ &= \frac{n_{e,0}\sigma_e c}{H_0}\left[-\frac{2[\Omega_{m,0}+(1-\Omega_{m,0})a^3]^{1/2}}{3\Omega_{m,0}a^{3/2}}\right]_{a_*}^{a_0} && (24)\\ &= \frac{2n_{e,0}\sigma_e c}{3H_0\Omega_{m,0}}\left[\left(\frac{\Omega_{m,0}}{a_*^3}+(1-\Omega_{m,0})\right)^{1/2}-1\right]. && (25)\end{aligned}$$

The integral was done with the aid of the Wolfram Mathematica online integrator (it has a simple closed form, somewhat to my surprise) and the final line uses $a_0 = 1$. The value of a_* for a given value of τ is

$$\begin{aligned} a_* &= \left[\frac{1}{\Omega_{m,0}}\left(\left(\frac{3H_0\Omega_{m,0}\tau}{2n_{e,0}\sigma_e c}+1\right)^2-(1-\Omega_{m,0})\right)\right]^{-1/3} && (26)\\ &\simeq \frac{1}{\Omega_{m,0}^{1/3}}\left(\frac{2n_{e,0}\sigma_e c}{3H_0\tau}\right)^{2/3}. && (27)\end{aligned}$$

The approximation holds if the quantity in parentheses is small, as it is for this problem. Using Ryden equation (6.29) for the relation between a and t for $a \ll a_{m\Lambda}$, which is again a good approximation for $\tau \simeq 1$, yields

$$t_* = \frac{1}{\Omega_{m,0}} \left(\frac{2}{3H_0}\right)^2 \frac{n_{e,0}\sigma_e c}{\tau}. \tag{28}$$

This is the same as equation 16 using $t_0 = 2/(3H_0)$, except for the factor $1/\Omega_{m,0}$. Plugging in the previous values and $\Omega_{m,0} = 0.3$ yields $t_* = 4.0 \times 10^7$ yrs and $z_* = 55$. Observations of the CMB by WMAP show that the optical depth to the surface of last scattering is $\tau = 0.084 \pm 0.016$, which implies that reionziation occurred at $z_* \simeq 11$.

3. The solid line in the figure on the next page is a plot of the model CMB power spectrum produced by CMBFAST for the default set of parameters. Some reading of the documentation confirmed that that the quantity in the second column of the CMBFAST output is $\ell(\ell+1)C_\ell/2\pi$ and so can be compared easily with the published CMB power spectrum. I actually used a maximum ℓ of 2000 (the default) instead of 1500.

The pattern of peak heights is very similar to that for the power spectrum derived from the WMAP data. The first peak is highest, the next two are lower but of nearly equal height, and subsequent peaks decrease in height. The locations of the peaks also agree well, though this was harder to tell because of the peculiar x-axis in the published plots of the WMAP power spectrum. The axis has an approximately, but not exactly, logarithmic scale that was chosen to emphasize the peaks and is actually never described in the papers. The locations of the first three peaks in the CMBFAST output are $\ell = 221$, 538, and 817. Finally, the heights of the peaks in the CMBFAST output is also similar to those for the real sky. Multiplying the heights output by $(2.725 \times 10^6 \mu\text{K})^2$ yields a height for the first peak of 5900, which agrees well with the published plots.

4. The dashed line in the figure shows the effect on the power spectrum of changing $\Omega_{\Lambda,0}$ to 0.4, while keeping all of the other parameters the same. The dotted line shows the power spectrum resulting from also changing $\Omega_{cdm,0}$ to 0.554 so that the model has the same flat geometry as the default case.

Changing $\Omega_{\Lambda,0}$ to 0.4 shifts all of the peaks in the power spectrum to larger ℓ compared to the default case. The locations of the first three peaks are at $\ell = 326$, 792, and 1203. These locations are all close to $1.47\times$ those of the default case. The height of the entire power spectrum is reduced, but the relative heights of the peaks remain very similar to what they were before. The results of the first numerical assignment showed that reducing $\Omega_{\Lambda,0}$ caused the angular diameter of the horizon at decoupling to become smaller. This agrees with the results from CMBFAST, which show the location of the first peak, which is approximately equal to that angular diameter, shifting to larger ℓ, hence, smaller angle. The fractional change in the location of the first peak is about -30%, which is somewhat smaller than the approximately -40% change I would expect on the basis of the numerical assignment. (Though one

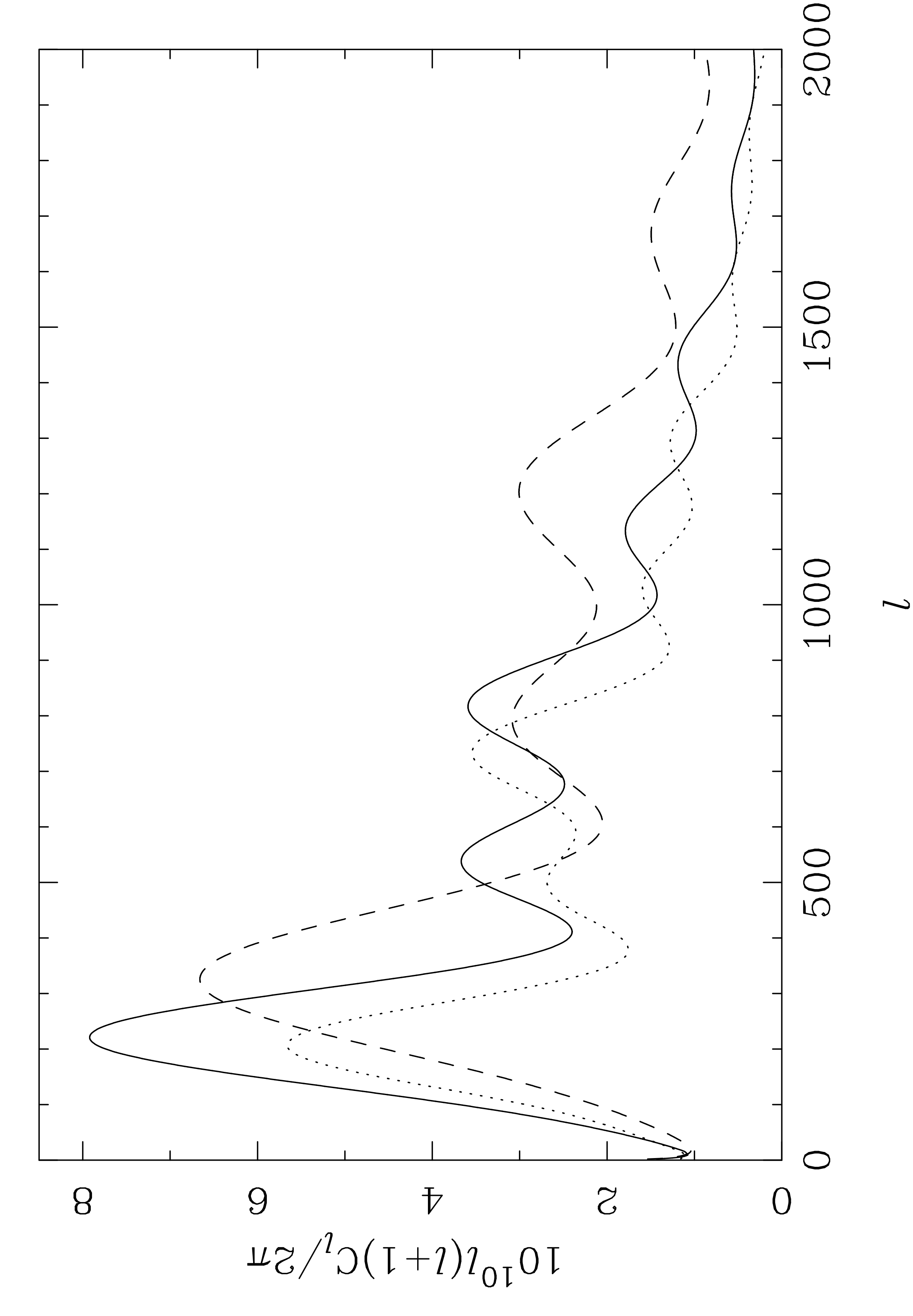

Figure 1: Power spectra produced by CMBFAST. The solid line is for the default set of parameters. The dashed lines changes $\Omega_{\Lambda,0}$ to 0.4. The dotted line also changes $\Omega_{cdm,0}$ to 0.554 so that the model has the same flat geometry as the default case.

should really recalculate $\Delta\theta_{hor}/\theta_{hor}$ for the same input parameters to make a careful comparison.)

Decreasing $\Omega_{\Lambda,0}$ to 0.4 but increasing $\Omega_{cdm,0}$ to 0.554 so that the model has the same flat geometry as the default case causes smaller changes in the locations of the peaks. The peaks shift to slightly lower ℓ's, with the first three being at $\ell = 206$, 500, and 734. The ratio of these locations to those in the default case is not as uniform as for the previous change, but are close to 0.9. The heights of the peaks are reduced compared to the default case and the relative heights are broadly similar, though now the third peak has been enhanced relative to the others. The first numerical assignment predicted that θ_{hor} decreased when $\Omega_{\Lambda,0}$ was decreased while keeping the geometry flat. This agrees in direction with the change in the location of the first peak predicted by CMBFAST. However, the predicted fractional change of about 14% is larger than the $\sim$7% fractional change predicted by CMBFAST.

The change in the location of the first peak when $\Omega_{\Lambda,0}$ is changed but the geometry is kept flat is much smaller than when the geometry is allowed to change. This supports the assertion in the text that the location of the first peak is primarily dependent on the curvature of the universe.