Ph 444 Solutions for Problem Set 6

1. (Ryden 7.5) The flux, f, received from a standard candle of luminosity L is (Ryden equation 7.21)

$$f = \frac{L}{4\pi d_L^2}, \tag{1}$$

where d_L is the luminosity distance. The angular diameter, $\delta\theta$, of a standard yardstick of size ℓ is (Ryden equation 7.33)

$$\delta\theta = \frac{\ell}{d_a}, \tag{2}$$

where d_a is the angular diameter distance. Thus, the surface brightness, Σ, of an object that is both a standard candle and a standard yardstick is

$$\Sigma \propto \frac{f}{(\delta\theta)^2} = \frac{L}{4\pi\ell^2}\left(\frac{d_a}{d_L}\right)^2. \tag{3}$$

Now from Ryden equation (7.37), $d_a = d_L/(1+z)^2$, so

$$\Sigma \propto \frac{L}{\ell^2}\left(\frac{1}{1+z}\right)^4. \tag{4}$$

Note that surface brightness decreases quickly with increasing redshift.

Observations of the surface brightness of these objects as a function of redshift cannot determine q_0 because the ratio d_a/d_L has no dependence on the cosmological model. Of course, measuring either angular diameters or fluxes as a function of redshift does constrain the model. But doing both provides no additional information.

2. This problem explores what is required to measure the equation of state of the dark energy using observations of type Ia supernovae. What we observe is the peak apparent magnitude, m, of each supernova. The distance modulus is then calculated from the corrected peak absolute magnitude of the supernova. The distance modulus to an individual type Ia supernova can be measured with an accuracy of $\sigma_{sn} = 0.15$ mag. This uncertainty comes mostly from our inability to exactly correct for the intrinsic spread of supernovae peak luminosities rather than the measurement uncertainty in the peak apparent magnitude of a supernova. The basis for using supernovae to probe cosmology is the relation between distance modulus and redshift and this problem adopts the form given by Ryden equation (7.52):

$$m - M = 43.17 - 5\log_{10}\left(\frac{H_0}{70\ \mathrm{kms^{-1}Mpc^{-1}}}\right) + 5\log_{10}(z) + 1.086(1-q_0)z. \tag{5}$$

Here q_0 is the deceleration constant which is given by Ryden equation (7.10):

$$q_0 = \frac{1}{2}\sum_w \Omega_{w,0}(1+3w). \tag{6}$$

This problem considers a universe that today contains primarily matter and quintessence with an equation of state parameter w. In this case,

$$q_0 = \frac{1}{2}\Omega_{m,0} + \frac{1}{2}\Omega_{q,0}(1+3w). \tag{7}$$

The first step is to calibrate the corrected supernova absolute magnitude, M, and the Hubble constant using supernovae at small redshift. For $z \ll 1$, the $5\log_{10}(z)$ term in equation 5 is much larger than the $1.086(1-q_0)z$ term and the relation between distance modulus and redshift is independent of the cosmological model. One can think of observations at small redshift determining the constant

$$C \equiv M - M_* + 43.17 - 5\log_{10}\left(\frac{H_0}{70\ \mathrm{kms^{-1}Mpc^{-1}}}\right), \tag{8}$$

where M_* is an arbitrary estimate of M. For observations of a set of $N/2$ supernovae, the uncertainty in C will be $\sigma_C = \sigma_{sn}/\sqrt{N/2}$.

At larger z, the relation between distance modulus and redshift does depend on q_0 and, hence, on the w of the dark energy. This is shown in Ryden Figure 7.5. Thus, observations of the distance moduli given by

$$m - M_* = C + 5\log_{10}(z) + 1.086(1-q_0)z \tag{9}$$

can determine q_0 and, hence, w.

How many supernovae must be observed to determine w to ± 0.01? The uncertainty in w is given by

$$\sigma_w^2 = \left(\frac{dw}{d(m-M_*)}\sigma_{(m-M_*)}\right)^2 + \left(\frac{dw}{dC}\sigma_C\right)^2, \tag{10}$$

since independent uncertainties add in quadrature. Using equation 5,

$$\frac{d}{dw}(m-M) = \frac{d}{dw}1.086(1-q_0)z = -1.086z\frac{dq_0}{dw}. \tag{11}$$

From equation 7,

$$\frac{dq_0}{dw} = \frac{d}{dw}\left[\frac{1}{2}\Omega_{m,0} + \frac{1}{2}\Omega_{q,0}(1+3w)\right] = \frac{3}{2}\Omega_{q,0}. \tag{12}$$

Thus,

$$\frac{dw}{d(m-M)} = \left(\frac{d(m-M)}{dw}\right)^{-1} = \left(-1.086z\frac{3}{2}\Omega_{q,0}\right)^{-1} = \left(-1.629z\Omega_{q,0}\right)^{-1}. \tag{13}$$

Similarly,

$$\frac{dw}{dC} = \left(-1.629z\Omega_{q,0}\right)^{-1}. \tag{14}$$

Thus,

$$\sigma_w = \frac{1}{1.629 z \Omega_{q,0}} \left(\sigma^2_{(m-M_*)} + \sigma^2_C\right)^{1/2} \tag{15}$$

$$= \frac{1}{1.629 z \Omega_{q,0}} \left(\left(\frac{\sigma_{sn}}{\sqrt{N/2}}\right)^2 + \left(\frac{\sigma_{sn}}{\sqrt{N/2}}\right)^2\right)^{1/2} \tag{16}$$

$$= \frac{2}{1.629 z \Omega_{q,0}} \left(\frac{\sigma_{sn}}{\sqrt{N}}\right). \tag{17}$$

If $z = 0.5$ and $\Omega_{q,0} = 0.7$, then $\delta w = 0.01$ requires that $N = 2.8 \times 10^3$. Since σ_w decreases linearly with z, only about one quarter as many supernovae would be required at $z = 1.0$. However, these more distant supernovae are more difficult to discover and observe. They are also more likely to suffer from systematic errors due to changing properties of the population of supernovae.

3. (Ryden 8.1) The average dark matter density interior to the Galactic orbit of the Sun is $\rho_{dm} = 0.04\ \mathrm{M_\odot\ pc^{-3}}$ (see Ryden equation 8.14). Adopt this as the density of dark matter in the solar neighborhood. This is somewhat too high, but by only $\times 3$ if the dark matter density profile is proportional to $1/r^2$ as expected for a flat rotation curve. The error is actually smaller since only about half of the mass inside the orbit of the Sun is dark matter. If the dark matter is composed of black holes or MACHOS with mass m_{dm}, then the number density of dark matter objects near the Sun is $n_{dm} = \rho_{dm}/m_{dm}$. A simple estimate of the distance to the nearest dark matter object is

$$\ell = n_{dm}^{-1/3} = \left(\frac{m_{dm}}{\rho_{dm}}\right)^{1/3} = \left(\frac{m_{dm}}{0.04\ \mathrm{M_\odot\ pc^{-3}}}\right)^{1/3} = (2.9\ \mathrm{pc}) \left(\frac{m_{dm}}{1\ \mathrm{M_\odot}}\right)^{1/3}. \tag{18}$$

To calculate the typical time for a dark matter object to pass within a distance s of the Sun, imagine the Sun moving with velocity v through a uniform density of stationary dark matter particles. The dark matter particles will be moving as well, probably with a comparable velocity, but we are only looking for an order-of-magnitude estimate. Then the time is just the time for a "collision" to occur with cross section πs^2. The average time between such collisions is about

$$T = \frac{1}{v n_{dm} \pi s^2} = \frac{m_{dm}}{v \rho_{dm} \pi s^2}. \tag{19}$$

For the velocity of the Sun, adopt its orbital velocity about the center of the Galaxy, $v = 220\ \mathrm{km\ s^{-1}} \simeq 220\ \mathrm{pc}\ (10^6\ \mathrm{yr})^{-1}$. Then for $s = 1\ \mathrm{AU} = 4.8 \times 10^{-6}$ pc, the time to approach within 1 AU is

$$T = \frac{m_{dm}}{(220\ \mathrm{pc}\ (10^6\ \mathrm{yr})^{-1})(0.04\ \mathrm{M_\odot\ pc^{-3}})\pi(4.8 \times 10^{-6}\ \mathrm{pc})^2} \tag{20}$$

$$= (1.5 \times 10^{15}\ \mathrm{yrs}) \left(\frac{m_{dm}}{1\ \mathrm{M_\odot}}\right). \tag{21}$$

Plugging in numbers gives the results in the table below. For the case of 10^{-8} $M_\odot$ black holes, the nearest would be 1.3×10^3 AU away. The encounter times are of interest because a sufficiently massive dark matter object passing within 1 AU of the Sun could strongly alter the Earth's orbit. Indeed, such a passage would perturb all of the planetary orbits, making the solar system unstable. We have no indications of large changes in the solar system since it formed 4.6 billion years ago. However, the encounter times for $m_{dm} > 10^{-3}$ $M_\odot$ are sufficiently long that this observation probably does not place any strong constraints on the mass of possible dark matter objects.

Table 1: Closest Dark Matter Object and Time to Approach Within 1 AU

m_{dm} ($M_\odot$)	n_{dm} (pc^{-3})	ℓ (pc)	T (yrs)
10^{-8}	4.0×10^6	6.3×10^{-3}	1.5×10^7
10^{-3}	40	0.29	1.5×10^{12}

4. (Ryden 8.3) The angular deflection due to gravitational lensing for an object that passes a distance R from a mass M is (Ryden equation 8.48)

$$\alpha = \frac{4GM}{c^2 R} \tag{22}$$

$$= \frac{4(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(6.0 \times 10^{24} \text{ kg})}{(2.998 \times 10^8 \text{ m s}^{-2})^2 (6.4 \times 10^6 \text{ m})} \tag{23}$$

$$\times \left(\frac{M}{6.0 \times 10^{24} \text{ kg}}\right)\left(\frac{6.4 \times 10^6 \text{ m}}{R}\right) \tag{24}$$

$$= (2.78 \times 10^{-9})\left(\frac{M}{6.0 \times 10^{24} \text{ kg}}\right)\left(\frac{6.4 \times 10^6 \text{ m}}{R}\right). \tag{25}$$

The deflection for a ray of light grazing the surface of the Earth, a white dwarf, and a neutron star are given in the table below. The effects of gravitational lensing in

Table 2: Angular Deflections

Object	M (kg)	R (m)	α
Earth	6.0×10^{24}	6.4×10^6	2.78×10^{-9} radians = 5.7×10^{-4} arcseconds
white dwarf	2.0×10^{30}	1.5×10^7	4.0×10^{-4} radians = 82 arcseconds
neutron star	3.0×10^{30}	1.2×10^4	0.74 radians = 43 degrees

the vicinity of a neutron star are large. This is not surprising, since the radius of a neutron star is only slightly larger than the event horizon of a black hole with the same mass.