Ph 444 Solutions for Problem Set 5

1. (Ryden 5.4) The proper distance today to a galaxy with redshift z in a flat, single-component universe with an equation of state defined by the parameter w is given by equation (5.54) from Ryden:

$$d_p(t_0) = \frac{c}{H_0}\left(\frac{2}{1+3w}\right)\left[1-(1+z)^{-(1+3w)/2}\right]. \quad (1)$$

The proper distance at the time of emission of the light is

$$d_p(t_e) = \frac{a_e}{a_0}d_p(t_0) = \frac{d_p(t_0)}{1+z}. \quad (2)$$

If $H_0 = 70$ km s^{-1} Mpc^{-1}, then $c/H_0 = 4300$ Mpc. The proper distance for a galaxy with $z = 7$ in a flat radiation- ($w = 1/3$), matter- ($w = 0$), or Λ-dominated ($w = -1$) universe is given in the table below.

universe	$d_p(t_0)$	$d_p(t_e)$
radiation, $w = 1/3$	$\frac{c}{H_0}\left[\frac{z}{1+z}\right] = 3800$ Mpc	$\frac{c}{H_0}\left[\frac{z}{(1+z)^2}\right] = 470$ Mpc
matter, $w = 0$	$\frac{2c}{H_0}\left[1-\frac{1}{\sqrt{1+z}}\right] = 5500$ Mpc	$\frac{2c}{H_0(1+z)}\left[1-\frac{1}{\sqrt{1+z}}\right] = 690$ Mpc
Λ, $w = -1$	$\frac{cz}{H_0} = 3.0 \times 10^4$ Mpc	$\frac{c}{H_0}\left[\frac{z}{1+z}\right] = 3800$ Mpc

The proper distance to a high-redshift object is strongly dependent on the cosmological model. Note that some people continue to give their answers with far more significant figures than are justified by the input data (in this problem, H_0 given to two significant figures).

2. (Ryden 6.3) This problem considers a spatially-flat universe composed of matter ($w = 0$) and quintessence ($w = -1/2$). If the fraction of the energy density in the form of matter today is $\Omega_{m,0}$, then the fraction in the form of quintessence today is $\Omega_{Q,0} = 1 - \Omega_{m,0}$. The ratio of the energy densities at the time corresponding to the scale factor a is

$$\frac{\epsilon_m}{\epsilon_Q} = \frac{\epsilon_{m,0}a^{-3}}{\epsilon_{Q,0}a^{-3/2}} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)a^{-3/2}. \quad (3)$$

The scale factor when $\epsilon_m/\epsilon_Q = 1$ is then given by

$$1 = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)a_{mQ}^{-3/2} \Rightarrow a_{mQ} = \left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)^{2/3}. \quad (4)$$

The Friedmann equation for a flat universe containing matter and quintessence is given by Ryden equation (6.6) with $\Omega_Q/a^{3/2}$ replacing Ω_Λ and $\Omega_0 = 1$:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{m,0}}{a^3} + \frac{1-\Omega_{m,0}}{a^{3/2}}. \quad (5)$$

Since $H = (da/dt)/a$, multiplying the above equation by a^2 and rearranging to put $H_0 dt$ on one side and da times a function of a on the other and then integrating yields

$$\int_0^t H_0 dt = \int_0^a \frac{da}{[\Omega_{m,0}\, a^{-1} + (1-\Omega_{m,0})a^{1/2}]^{1/2}} \tag{6}$$

$$\Rightarrow H_0 t = \int_0^a \frac{a^{1/2} da}{[\Omega_{m,0} + (1-\Omega_{m,0})a^{3/2}]^{1/2}} \tag{7}$$

$$= \int_0^a \frac{(2/3) da^{3/2}}{[\Omega_{m,0} + (1-\Omega_{m,0})a^{3/2}]^{1/2}} \tag{8}$$

$$= \left[\frac{4\sqrt{\Omega_{m,0} + (1-\Omega_{m,0})a^{3/2}}}{3(1-\Omega_{m,0})}\right]\Bigg|_0^a \tag{9}$$

$$= \frac{4\sqrt{\Omega_{m,0} + (1-\Omega_{m,0})a^{3/2}}}{3(1-\Omega_{m,0})} - \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})} \tag{10}$$

$$= \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left[\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\, a^{3/2} + 1} - 1\right] \tag{11}$$

$$\Rightarrow H_0 t = \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left[\sqrt{\left(\frac{a}{a_{mQ}}\right)^{3/2} + 1} - 1\right]. \tag{12}$$

This expression can be inverted to give $a(t)$:

$$\frac{a(t)}{a_{mQ}} = \left[\left(\left(\frac{3(1-\Omega_{m,0})}{4\sqrt{\Omega_{m,0}}}\right) H_0 t + 1\right)^2 - 1\right]^{2/3} \tag{13}$$

or, eliminating a_{mQ}:

$$a(t) = \left[\left(\frac{\Omega_{m,0}}{1-\Omega_{m,0}}\right)\left(\left(\left(\frac{3(1-\Omega_{m,0})}{4\sqrt{\Omega_{m,0}}}\right) H_0 t + 1\right)^2 - 1\right)\right]^{2/3}. \tag{14}$$

The limits for small and large a are easiest to see from equation 12. If $a \ll a_{mQ}$, then

$$H_0 t \approx \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left[1 + \frac{1}{2}\left(\frac{a}{a_{mQ}}\right)^{3/2} - 1\right] = \frac{2\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left(\frac{a}{a_{mQ}}\right)^{3/2} \tag{15}$$

$$\Rightarrow a \approx \left(\frac{3\sqrt{\Omega_{m,0}}}{2}(H_0 t)\right)^{2/3}. \tag{16}$$

The last step uses the definition of a_{mQ}. This is the time dependence for a flat, matter-dominated universe, as it should be in this limit. If $a \gg a_{mQ}$, then

$$H_0 t \approx \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left[\sqrt{\left(\frac{a}{a_{mQ}}\right)^{3/2}}\right] = \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left(\frac{a}{a_{mQ}}\right)^{3/4} \tag{17}$$

$$\Rightarrow a \approx \left(\frac{3\sqrt{1-\Omega_{m,0}}}{4}(H_0 t)\right)^{4/3}. \tag{18}$$

This is the dependence of a on t for a flat, quintessence-dominated universe.

The current age of the universe comes from evaluating equation 12 at $a = 1$:

$$H_0 t_0 = \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left[\sqrt{\frac{1-\Omega_{m,0}}{\Omega_{m,0}}+1}-1\right] \tag{19}$$

$$= \frac{4\sqrt{\Omega_{m,0}}}{3(1-\Omega_{m,0})}\left[\sqrt{\frac{1}{\Omega_{m,0}}}-1\right] \tag{20}$$

$$= \frac{4(1-\sqrt{\Omega_{m,0}})}{3(1-\Omega_{m,0})} \tag{21}$$

$$\Rightarrow t_0 = \frac{4}{3(1+\sqrt{\Omega_{m,0}})}\frac{1}{H_0}. \tag{22}$$

In the limit $\Omega_{m,0} \to 0$, $t_0 \to 4/(3H_0)$. Similarly, in the limit $\Omega_{m,0} \to 1$, $t_0 \to 2/(3H_0)$. These limits agree with equation (5.44) of Ryden, which gives t_0 for single-component flat universes. It is easy to show that t_0 decreases with increasing $\Omega_{m,0}$ for values between zero and one.

3. Complete solution derived in class.