Ph 444 Solutions for Problem Set 3

1. The critical energy density is $\epsilon_c = 3H_0^2/(8\pi G c^2)$ and so the critical density is $\rho_c = 3H_0^2/(8\pi G)$. The uncertainty in the critical density is

$$\begin{aligned} \sigma_{\rho_c} &= \frac{\partial \rho_c}{\partial H_0}\sigma_{H_0} = \frac{\partial}{\partial H_0}\left(\frac{3H_0^2}{8\pi G}\right)\sigma_{H_0} = \left(\frac{6H_0}{8\pi G}\right)\sigma_{H_0} && (1)\\ \Rightarrow \frac{\sigma_{\rho_c}}{\rho_c} &= 2\frac{\sigma_{H_0}}{H_0} && (2)\end{aligned}$$

I find that fractional uncertainties are usually more meaningful than the uncertainty itself. The formulae are also often simpler.

Plugging in the value of H_0 gives

$$\rho_c = \frac{3(70.6 \text{ km s}^{-1}\text{Mpc}^{-1})^2(1 \text{ Mpc}/3.086\times 10^{19} \text{ km})^2}{8\pi(6.673\times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-1})} = 9.362\times 10^{-27} \text{ kg m}^{-3}. \quad (3)$$

For use in problem 2, convert the units to $M_\odot$ Mpc^{-3}:

$$\rho_c = \frac{(3.086\times 10^{22} \text{ m/Mpc})^3}{1.989\times 10^{30} \text{ kg}/M_\odot}(9.362\times 10^{-27} \text{ kg m}^{-3}) = 1.38\times 10^{11}\ M_\odot \text{ Mpc}^{-3}. \quad (4)$$

The fractional uncertainty is

$$\frac{\sigma_{\rho_c}}{\rho_c} = 2\left(\frac{1.8 \text{ km s}^{-1}\text{Mpc}^{-1}}{70.6 \text{ km s}^{-1}\text{Mpc}^{-1}}\right) = 2(0.025) = 0.051. \quad (5)$$

Thus, the critical density is

$$\rho_c = (9.362\pm 0.48)\times 10^{-27} \text{ kg m}^{-3} = (1.38\pm 0.07)\times 10^{11}\ M_\odot \text{ Mpc}^{-3}. \quad (6)$$

2. If j is the luminosity density of galaxies, then the average M/L per galaxy needed for galaxies to produce the critical density is $(M/L)_c = \rho_c/j$. The propagation of errors is slightly tricky in this case as both j and ρ_c depend on H_0 and so errors in these quantities are not independent. Adding uncertainties in quadrature requires that the uncertainties be independent. The solution is to express $(M/L)_c$ explicitly as a function of H_0.

$$\begin{aligned} \left(\frac{M}{L}\right)_c &= \frac{\rho_c}{j} = \left(\frac{3H_0^2}{8\pi G}\right)\left(\frac{1}{j_h(H_0/100 \text{ km s}^{-1}\text{Mpc}^{-1})}\right) && (7)\\ &= \left(\frac{3(100 \text{ km s}^{-1}\text{Mpc}^{-1})}{8\pi G}\right)\left(\frac{H_0}{j_h}\right). && (8)\end{aligned}$$

Then the usual propagation of errors formula gives

$$\sigma_{(M/L)_c} = \left[\left(\frac{\partial(M/L)_c}{\partial H_0}\sigma_{H_0}\right)^2 + \left(\frac{\partial(M/L)_c}{\partial j_h}\sigma_{j_h}\right)^2\right]^{1/2} \tag{9}$$

$$\Rightarrow \frac{\sigma_{(M/L)_c}}{(M/L)_c} = \left[\left(\frac{\sigma_{H_0}}{H_0}\right)^2 + \left(\frac{\sigma_{j_h}}{j_h}\right)^2\right]^{1/2} \tag{10}$$

Plugging in numbers gives

$$\left(\frac{M}{L}\right)_c = \frac{\rho_c}{j} \tag{11}$$

$$= \frac{1.38 \times 10^{11}\ M_\odot\ \mathrm{Mpc}^{-3}}{(1.986 \times 10^8\ L_\odot\ \mathrm{Mpc}^{-3})(70.6\ \mathrm{km\ s}^{-1}\mathrm{Mpc}^{-1}/100\ \mathrm{km\ s}^{-1}\mathrm{Mpc}^{-1})} \tag{12}$$

$$= 9.842 \times 10^2\ M_\odot\ {L_\odot}^{-1} \tag{13}$$

and

$$\frac{\sigma_{(M/L)_c}}{(M/L)_c} = \left[\left(\frac{1.8\ \mathrm{km\ s}^{-1}\mathrm{Mpc}^{-1}}{70.6\ \mathrm{km\ s}^{-1}\mathrm{Mpc}^{-1}}\right)^2 + \left(\frac{0.031 \times 10^8\ L_\odot \mathrm{Mpc}^{-3}}{1.986 \times 10^8\ L_\odot \mathrm{Mpc}^{-3}}\right)^2\right]^{1/2} \tag{14}$$

$$= \left[(0.025)^2 + (0.016)^2\right]^{1/2} = 0.030. \tag{15}$$

Thus, $(M/L)_c = (9.842 \pm 0.30) \times 10^2\ M_\odot\ L_\odot^{-1}$. This value is much larger than the $M/L = 2$ (in solar units) that is typical for stellar populations. If galaxies provide the critical density, most of their mass must be in a form other than stars.

3. (Ryden 4.4) If the mass of a baseball is m_{bb}, then the number density of baseballs required to produce the critical density is

$$n = \frac{\rho_c}{m_{bb}} = \frac{9.36 \times 10^{-27}\ \mathrm{kg\ m}^{-3}}{0.145\ \mathrm{kg}} \tag{16}$$

$$= 6.46 \times 10^{-26}\ \mathrm{m}^{-3} = 1.90 \times 10^{42}\ \mathrm{Mpc}^{-3}. \tag{17}$$

Here I have used the value of the critical density from problem 1. Your line of sight would intersect a baseball at an average distance of

$$\mathrm{mfp} = \frac{1}{n\pi r_{bb}^2} = \frac{1}{(6.46 \times 10^{-26}\ \mathrm{m}^{-3})\pi(0.0369\ \mathrm{m})^2} \tag{18}$$

$$= 3.62 \times 10^{27}\ \mathrm{m} = 1.17 \times 10^5\ \mathrm{Mpc}. \tag{19}$$

This mean free path is much larger the the approximate size of the visible universe, $c/H_0 \simeq 4000$ Mpc. Thus, even if the universe had a critical density of baseballs, they would cause negligible absorption of light. So the transparency of the universe does not place a significant constraint on the mass density in baseballs. Such macroscopic objects cause very little absorption for their mass and are very hard to detect. Fortunately, we will see that observations and the theory of primordial nucleosynthesis require that most of the baryonic matter in the universe be in the form of hydrogen and helium. These elements do not form solids under interstellar conditions.

4. (Ryden 4.5) This problem requires deriving the pressure for a collection of particles whose energy may be non-relativistic, very relativistic, or something in between. If m is the rest mass and p the momentum of each particle, then the energy per particle is $E = (m^2c^4 + p^2c^2)^{1/2}$. These particles are non-interacting and affected by no forces, so m is constant and p varies with the scale factor as $p = p_0(a_0/a)$. This last result was given in the problem.

One way to derive the pressure is to start with the First Law of Thermodynamics for some volume V (Ryden equation 4.32):

$$\dot{\mathcal{E}} + P\dot{V} = 0. \tag{20}$$

The energy in the volume is $\mathcal{E} = NE$, where N is the fixed number of particles in the volume. Then $\dot{\mathcal{E}} = N\dot{E}$. Thus,

$$P = -\frac{\dot{\mathcal{E}}}{\dot{V}} = -\frac{N\dot{E}}{\dot{V}} = -\frac{N(dE/da)(da/dt)}{(dV/da)(da/dt)} = -\frac{N(dE/da)}{dV/da}. \tag{21}$$

Now $V = V_0(a/a_0)^3$, so $dV/da = 3V_0a^2/a_0^3 = 3V/a$. Similarly,

$$\begin{aligned} \frac{dE}{da} &= \frac{d}{da}(m^2c^4 + p_0^2(a_0/a)^2c^2)^{1/2} && (22)\\ &= \frac{1}{2}\frac{p_0^2a_0^2(-2/a^3)c^2}{(m^2c^4 + p_0^2(a_0/a)^2c^2)^{1/2}} && (23)\\ &= -\frac{p^2c^2/a}{E}. && (24)\end{aligned}$$

Substituting the above expressions into Equation 21 yields

$$P = -\frac{N(-p^2c^2/(aE))}{3V/a} = \left(\frac{N}{V}\right)\frac{p^2c^2}{3E} = \frac{n(E^2 - m^2c^4)}{3E}, \tag{25}$$

where n is the number density of particles. This is the general expression for the pressure of a set of non-interacting particles with identical energies.

The same formula results from starting with the expression for pressure given in class, which uses the idea that pressure is the flux of momentum. This argues that $P = npv$ and correctly doing the angular integrals for an isotropic distribution of velocity vectors adds a factor of 1/3: $P = npv/3$. Using the formula from special relativity, $v/c = pc/E$, then yields Equation 25.

The equation-of-state parameter, w, is not a constant for the general case. It is the expression

$$w \equiv \frac{P}{\epsilon} = \frac{P}{nE} = \frac{E^2 - m^2c^4}{3E^2} = \frac{p^2c^2}{3(p^2c^2 + m^2c^4)}. \tag{26}$$

Thus, w depends on a as

$$w = \frac{(p_0/a)^2c^2}{3((p_0/a)^2c^2 + m^2c^4)} = \frac{p_0^2c^2}{3(p_0^2c^2 + m^2c^4a^2)}. \tag{27}$$

If the particles are ultra-relativistic, $E \gg mc^2$ and Equation 25 reduces to $P = nE/3 = \epsilon/3$. Thus, $w = 1/3$ in this limit. If the particles are non-relativistic, then $E \approx mc^2$ and $E^2 - m^2c^4 = p^2c^2 \approx m^2v^2c^2$. In this limit, $P = nmv^2/3 = (v/c)^2(nmc^2)/3 = (v/c)^2\epsilon/3$. Thus, $w = (v/c)^2/3 \approx 0$.