Ph 444 Solutions for Problem Set 1

1. (Ryden 2.2) To decide how far one can see on average in a universe filled with spherical objects of radius R, it is simplest to think of a long cylinder along the line of sight. If an object is closer than R to the line of sight, then the line of sight intersects its surface. For a distance ℓ, the cylindrical volume which would contain such objects is $\pi R^2 \ell$. If the density of objects is n, then the volume that will on average contain one object is defined by $\pi R^2 \ell n = 1$ and the average distance to which we see before our vision is blocked is

$$\ell = \frac{1}{\pi R^2 n}. \quad (1)$$

You have probably seen a similar formula before as the equation for the mean-free-path between collisions of, for example, atoms in a gas. Another way to derive Equation (1) is to calculate the probability of each mean-free-path when the probability of collision in each $d\ell$ along the line of sight is $d\ell \pi R^2$. Calculating the average value of ℓ using the probability distribution yields Equation (1).

For a density of stars equal to 10^9 Mpc^{-3} and a stellar radius of

$$R = (7 \times 10^8 \text{ m})/(3.086 \times 10^{22} \text{ m Mpc}^{-1}) = 2.3 \times 10^{-14} \text{ Mpc},$$

the average distance is

$$\ell = \frac{1}{\pi (2.3 \times 10^{-14} \text{ Mpc})^2 (10^9 \text{ Mpc}^{-3})} = 6.2 \times 10^{17} \text{ Mpc}.$$

This distance is much larger than the distance light has traveled since the Big Bang, about 6000 Mpc, which is a large part of the reason why the night sky is dark.

If galaxies have an average density of 1 Mpc^{-3} and an average radius of $R = 2.0 \times 10^3$ pc $= 2.0 \times 10^{-3}$ Mpc, then the distance to a galaxy along the typical line of sight is

$$\ell = \frac{1}{\pi (2.0 \times 10^{-3} \text{ Mpc})^2 (1 \text{ Mpc}^{-3})} = 8.0 \times 10^4 \text{ Mpc}.$$

Thus, we might expect to see a galaxy everywhere on the sky; however, this distance is large enough that the expansion and geometry of the universe must be taken into account.

2. (Ryden 2.4) Observations of one type of neutrino "oscillating" into another type yield two constraints on the masses of the electron, muon, and tau neutrinos ($m(\nu_e)$, $m(\nu_\mu)$, and $m(\nu_\tau)$, respectively):

$$(m(\nu_\mu)^2 - m(\nu_e)^2)c^4 = 5 \times 10^{-5} \text{ eV}^2 \quad (2)$$

$$(m(\nu_\tau)^2 - m(\nu_\mu)^2)c^4 = 3 \times 10^{-3} \text{ eV}^2. \quad (3)$$

Equations (2) and (3) obviously require that $m(\nu_e)$ be the smallest of the three, since the masses must be positive. With two constraints, the sum of the three masses can be written in terms of one of the masses, say $m(\nu_e)$. First, write $m(\nu_\mu)$ and $m(\nu_\tau)$ in terms of $m(\nu_e)$.

$$\begin{aligned} m(\nu_\mu)c^2 &= \sqrt{m(\nu_e)^2c^4 + 5\times 10^{-5}\ \mathrm{eV}^2} \\ m(\nu_\tau)c^2 &= \sqrt{m(\nu_\mu)^2c^4 + 3\times 10^{-3}\ \mathrm{eV}^2} = \sqrt{m(\nu_e)^2c^4 + 3.05\times 10^{-3}\ \mathrm{eV}^2} \end{aligned}$$

Then the sum of the masses is

$$\begin{aligned} (m(\nu_e) + m(\nu_\mu) + m(\nu_\tau))c^2 = m(\nu_e)c^2 &+ \sqrt{m(\nu_e)^2c^4 + 5\times 10^{-5}\ \mathrm{eV}^2} \\ &+ \sqrt{m(\nu_e)^2c^4 + 3.05\times 10^{-3}\ \mathrm{eV}^2}. \end{aligned}$$

All three terms of the sum decrease monotonically with $m(\nu_e)$ as long as $m(\nu_e)$ is positive (for a formal proof, take the derivative). Thus, the sum is minimized by making $m(\nu_e)$ zero (or, at least, much smaller than $m(\nu_\mu)$ and $m(\nu_\tau)$). To reach the same conclusion more formally, calculate $d(m(\nu_e) + m(\nu_\mu) + m(\nu_\tau))/dm(\nu_e)$ and set it equal to zero. The result is a negative value for $m(\nu_e)$, which is unphysical.

Thus, the neutrino masses which minimize their sum are

$$\begin{aligned} m(\nu_e)c^2 &= 0\ \mathrm{eV} \\ m(\nu_\mu)c^2 &= 7.1\times 10^{-3}\ \mathrm{eV} \quad (1.3\times 10^{-38}\ \mathrm{kg}) \\ m(\nu_\tau)c^2 &= 5.5\times 10^{-2}\ \mathrm{eV} \quad (9.8\times 10^{-38}\ \mathrm{kg}). \end{aligned}$$

The typical energy of a CMB photon today is 6.3×10^{-4} eV. Since the universe is thought to contain about as many of each species of neutrino per cubic centimeter as CMB photons (there should actually be 4/11 as many neutrinos as photons), the energy density in neutrinos is larger than the energy density of photons.

3. (Ryden 2.5) If photons lose energy as they travel according to the equation

$$\frac{dE}{dr} = -KE, \tag{4}$$

then

$$\frac{dE}{E} = -K dr$$

and integration of the left-hand side between E_0 and $E(r)$ and the right-hand side between 0 and r yields $E(r) = E_0 \exp(-Kr)$.

The frequency and energy of a photon are related through Planck's constant, $E = h\nu$, so the relation between the energy and the wavelength is $E = hc/\lambda$. Thus, in the tired light hypothesis the wavelength of light increases with the distance traveled as $\lambda(r) = \lambda_0 \exp(Kr)$. This equation implies a relation between the redshift, z, and the distance travel of

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}} = \frac{\lambda(r) - \lambda_0}{\lambda_0} = e^{Kr} - 1. \tag{5}$$

In the limit $Kr \ll 1$ (hence, $z \ll 1$), Equation (5) reduces to $z = Kr$ to first order. Since $z = (H_0/c)r$ (Ryden eq. 2.5), the tired light hypothesis requires $K = H_0/c = 1/(4300\ \text{Mpc})$, where I have used the value for c/H_0 from page 16 of Ryden. Slightly more explicitly, $K = 2.3 \times 10^{-4}\ \text{Mpc}^{-1}$.

4. The goals of this problem were to examine some real data on the number of galaxies per square degree as a function of limiting magnitude and to explore how a change of galaxy luminosity with distance (*i.e.*, with time) could affect these number counts.

a) We derived in class that for an homogeneous Euclidean universe with no expansion, the number of galaxies per square degree, N, varies as a function of limiting magnitude, m, as $10^{0.6m}$. Thus, the Euclidean slope in a plot of $\log(N)$ *vs.* m is 0.6. This is the dashed line in the figure which accompanied this problem.

I estimated that the counts from the Millennium Galaxy Catalog (the open circles in the figure) began to depart significantly from the dashed line at the apparent magnitude $B_{MGC} = 19$. Values in the range $18 - 20$ are still reasonable, as the point of first departure is somewhat subjective.

The relation between the distance modulus and distance is

$$m - M = 5\log(d/(10\ \text{pc})) \Rightarrow d = (10\ \text{pc})10^{0.2(m-M)}. \tag{6}$$

If $m = B_{MGC} = 19$ and $M_B = -19.5$, then

$$d = (10\ \text{pc})10^{0.2(19-(-19.5))} = 5.0 \times 10^8\ \text{pc} = 500\ \text{Mpc}.$$

This is a large distance, but much smaller than the Hubble distance of $c/H_0 = 4300$ Mpc.

b) I drew a line through the MGC data points with magnitudes between 19 and 24. This line also went through the number counts at fainter magnitudes from other surveys reasonably well. This line had a slope of

$$\frac{\Delta \log(N)}{\Delta m} = \frac{5.32 - (-0.39)}{27 - 14} = 0.44. \tag{7}$$

c) This part of the problem asked you to rederive the number-count relation for a universe in which the galaxy density remains constant, but galaxy luminosities vary with distance as $L(r) = L_0(r/r_0)^a$. With this dependence of luminosity on distance, the flux received at Earth from a galaxy at distance r is

$$f = \frac{L(r)}{4\pi r^2} = \frac{L_0(r/r_0)^a}{4\pi r^2} = \frac{L_0}{4\pi r_0^a}\frac{1}{r^{2-a}}. \tag{8}$$

Then the galaxies with a flux larger than f_ℓ must be closer than the distance

$$r_\ell = \left(\frac{L_0}{4\pi r_0^a}\frac{1}{f_\ell}\right)^{1/(2-a)}. \tag{9}$$

The number of galaxies with a flux larger than f_ℓ is

$$N(> f_\ell) = \frac{4\pi}{3} r_\ell^3 n = \frac{4\pi}{3} n \left(\frac{L_0}{4\pi r_0^a} \frac{1}{f_\ell} \right)^{3/(2-a)} \propto f_\ell^{-3/(2-a)}, \tag{10}$$

where n is the number density of galaxies.

The relation between flux and apparent magnitude is $m = -2.5 \log(f) + constant$, so $f \propto 10^{-0.4m}$. Substituting this last relation into Equation (10) shows that the number of galaxies brighter than the limiting magnitude m increases as

$$N(< m) \propto 10^{1.2/(2-a)}. \tag{11}$$

The plot in the figure for this problem actually shows $dN(< m)/dm$, but, since the derivative of an exponential is the same exponential, the slope of the relation in a $\log(dN/dm)$ *vs.* m plot is the same as that given by Equation (11). Thus, the slope of 0.44 implies that $0.44 = 1.2/(2 - a)$ or $a = -0.73$. In this simple toy model, the galaxy number counts imply that galaxies get fainter with increasing distance beyond a distance of about 500 Mpc. For distances not much larger than 500 Mpc, this change is probably caused by the redshifting of galaxy emission out of the blue bandpass of the B_{MGC} magnitudes. Galaxies are not very luminous at ultraviolet wavelengths. At still larger distances, the number-count relation must include the effect of the expanding (and possibly curved) universe on the observations. The solid line in the figure, labeled "2dFGRS no evolution" shows the prediction of such a calculation. For very faint galaxies, say $B_{MGC} > 22$, the number of galaxies per square degree is larger than the "no evolution" prediction. This argues that galaxies were either more luminous or more numerous in the early universe. It is impossible to distinguish between these two possibilities from number counts alone. Studies that use redshifts to measure the distances of galaxies and so determine the galaxy luminosity function at different distances have shown that the excess galaxy counts are caused by galaxies being more luminous at earlier times.

Number counts have also been applied to quasars, which are more luminous than galaxies and so can be seen at larger distances. In this case it turns out that changes in both quasar luminosity and (comoving) number density contribute to the observed relation. Quasars were more common in the early universe, with the comoving density peaking between redshift 2 and 3 (for example, see Gordon *et al.*, 2006, AJ, 131, 2766, "The Sloan Digital Sky Survey Quasar Survey: Quasar Luminosity Function from Data Release 3").