Ph 444 Solutions for Numerical Assignment 1

1. The particle horizon at the time t is calculated by taking each interval of proper distance, cdt', covered by a photon between t' and $t' + dt'$, increasing it by the expansion of the universe between t' and t, $a(t)/a(t')$, and adding up the intervals between zero and t:

$$d_{hor}(t) = a(t) \int_0^t \frac{cdt'}{a(t')}. \tag{1}$$

Changing variables in the integral to a' and using $da' = \dot{a}'dt$, $H = \dot{a}/a$, and Ryden equation 6.6 for H:

$$\begin{aligned}
d_{hor}(t) &= a(t) \int_0^{a(t)} \frac{cda'}{a'\dot{a}'} = a(t) \int_0^{a(t)} \frac{cda'}{(a')^2 H(a)} && (2)\\
&= \frac{ca(t)}{H_0} \int_0^{a(t)} \frac{da'}{(a')^2 \left(\Omega_{r,0}/(a')^4 + \Omega_{m,0}/(a')^3 + \Omega_{\Lambda,0} + (1-\Omega_0)/(a')^2\right)^{1/2}} && (3)\\
&= \frac{c}{(1+z)H_0} \int_0^{1/(1+z)} \frac{da'}{\left(\Omega_{r,0} + \Omega_{m,0}a' + \Omega_{\Lambda,0}(a')^4 + (1-\Omega_0)(a')^2\right)^{1/2}} && (4)
\end{aligned}$$

The last step uses $a(t) = 1/(1+z)$.

The dimensionless integral in equation 4 does not have a simple closed form. It can be expressed in terms of exotic special functions, but this is not of much use in obtaining numerical values. I wrote a simple program to evaluate the integral using trapezoidal rule integration with 10^4 equal-size steps in a' (*i.e.*, multiplying the integrand evaluated at the middle of the interval by the interval width and summing the contribution from all of the intervals). I chose the number of steps by increasing the number until the result stopped changing. The value of the integral, I, for $\Omega_{r,0} = 8.4 \times 10^{-5}$, $\Omega_{m,0} = 0.27$, $\Omega_{\Lambda,0} = 0.73$, and $z = 1090$ is $I = 6.697 \times 10^{-2}$. Since $c/H_0 = 4220$ Mpc for $H_0 = 71$ km/s/Mpc, $d_{hor}(z = 1090) = 0.2592$ Mpc. This small size reflects the young age of the universe at the time of decoupling, about 300,000 yrs.

The interval of comoving coordinate that corresponds to the size of the horizon at $z = 1090$ is $r = d_{hor}(z = 1090)/a(z = 1090)$. The distance today that corresponds to this interval of comoving coordinate is just $a_0 r = d_{hor}(z = 1090)(a_0/a(z = 1090)) = d_{hor}(1 + z) = 283$ Mpc. Thus, any structure imposed on the size of the horizon at the time of decoupling will show up at large physical sizes today, though this size is still much smaller than the particle horizon today.

For $z = 1090$, it is a good approximation to ignore the $\Omega_{\Lambda,0}(a')^4$ and $(1-\Omega_0)(a')^2$ terms in the denominator of the integrand since they are smaller by a factor of at least 10^3 compared to the $\Omega_{m,0}a'$ term, even at the upper limit of the integration. The integral then can be expressed in a simple form:

$$I = \frac{2}{\Omega_{m,0}} \left(\left(\Omega_{r,0} + \frac{\Omega_{m,0}}{1+z} \right)^{1/2} - \left(\Omega_{r,0}\right)^{1/2} \right) \tag{5}$$

Plugging in the standard values yields $I = 6.697 \times 10^{-2}$, the same as the numerical evaluation. This approximation could have been used for parts 2 and 3.

2. Changing $\Omega_{r,0}$, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$ by ±10% in turn produces the values of d_{hor} in the second column of Table 1. The third column is the fractional change, $\Delta d_{hor}/d_{hor} \equiv (d_{hor}(new) - d_{hor}(old))/d_{hor}(old)$. Changing $\Omega_{\Lambda,0}$ has very little effect on d_{hor} because the cosmological constant contributes significantly to the energy density only at much later times than $z = 1090$. Changing $\Omega_{r,0}$ and $\Omega_{m,0}$ produce changes of similar size because the universe becomes matter dominated only shortly before the epoch of decoupling. I did not ask for an explanation of the signs of the changes in d_{hor}. However, increasing the energy density of either radiation or matter in the universe at all expansion factors (by increasing $\Omega_{r,0}$ or $\Omega_{m,0}$) shortens the age of the universe at each expansion factor (the evolution time of a gravitationally bound system is always proportional to $1/\sqrt{G\rho}$) and the shorter age implies a smaller particle horizon.

3. Increasing $\Omega_{\Lambda,0}$ by 10% while decreasing $\Omega_{m,0}$ so as to keep $\Omega_0 = 1.0$ (*i.e.*, $\Omega_{\Lambda,0} = 0.803$ and $\Omega_{m,0} = 0.197$) yields the values for d_{hor} and $\Delta d_{hor}/d_{hor}$ given near the end of Table 1. The following line gives the values for a corresponding 10% decrease of $\Omega_{\Lambda,0}$. The fractional change in d_{hor} for this case is the largest in the column because of the large fractional change in $\Omega_{m,0}$.

Table 1: Particle Horizon, Angular Diameter Distance, and Angular Size

parameters	d_{hor} (Mpc)	$\Delta d_{hor}/d_{hor}$	d_a (Mpc)	$\Delta d_a/d_a$	θ_{hor} (deg)	$\Delta\theta_{hor}/\theta_{hor}$
standard	0.2592		12.833		1.157	
$\Omega_{r,0} + 10\%$	0.2530	-0.024	12.826	-4.9×10^{-4}	1.130	-0.024
$\Omega_{r,0} - 10\%$	0.2661	$+0.026$	12.839	$+4.9 \times 10^{-4}$	1.187	$+0.026$
$\Omega_{m,0} + 10\%$	0.2531	-0.024	11.87	-0.075	1.222	$+0.055$
$\Omega_{m,0} - 10\%$	0.2660	$+0.026$	13.98	$+0.090$	1.090	-0.058
$\Omega_{\Lambda,0} + 10\%$	0.2592	$+3.3 \times 10^{-5}$	11.35	-0.116	1.309	$+0.131$
$\Omega_{\Lambda,0} - 10\%$	0.2592	-3.4×10^{-5}	14.27	$+0.112$	1.041	-0.101
Keeping $\Omega_0 = 1.0$:						
$\Omega_{\Lambda,0} + 10\%$	0.2790	$+0.076$	14.52	$+0.131$	1.101	-0.049
$\Omega_{\Lambda,0} - 10\%$	0.2436	-0.060	11.67	-0.091	1.197	$+0.034$
With $w = -0.5$:						
$\Omega_{Q,0} = 0.73$	0.2592	-9×10^{-6}	11.77	-0.083	1.262	$+0.090$

4. The angular diameter distance is (Ryden equation 7.9)

$$d_a = \frac{S_k(r)}{(1+z)}, \tag{6}$$

where

$$S_k(r) = \begin{cases} R_0 \sin(r/R_0) & (k = +1) \\ r & (k = 0) \\ R_0 \sinh(r/R_0) & (k = -1). \end{cases} \tag{7}$$

The radius of curvature R_0 is given by Ryden equation (4.31):

$$\frac{k}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1) \Rightarrow \frac{1}{R_0} = \frac{H_0}{c}\sqrt{|\Omega_0 - 1|}. \tag{8}$$

The coordinate distance r covered by a photon as it travels from the object to us is given by

$$\begin{aligned} r &= \int_{t_e}^{t_0} \frac{cdt'}{a(t')} \qquad (9)\\ &= \int_{a_e}^{a_0} \frac{cda'}{(a')^2 H(a)} \qquad (10)\\ &= \frac{c}{H_0}\int_{1/(1+z)}^{1} \frac{da'}{\left(\Omega_{r,0} + \Omega_{m,0}a' + \Omega_{\Lambda,0}(a')^4 + (1-\Omega_0)(a')^2\right)^{1/2}}. \qquad (11)\end{aligned}$$

The dimensionless integral in the expression for r is the same as in that for d_{hor} except for the different limits. I again evaluated the integral with the trapezoidal rule and 10^4 steps. However, I decided to use equal numbers of steps per $\ln(a)$ in order to improve the accuracy resulting from changes in $\Omega_{r,0}$. This term is only important for small a'. Thus, I changed the variable in the integral to $x = -\ln(a') \Rightarrow dx = -da'/a'$. The range of the integral is then 0 to $\ln(1+z)$.

Ryden notes (equation 7.38) that $d_a = d_p(t_0)/(1+z)$ if the universe is flat, $k = 0$, and many in the class used this form. Equation (7.41) also states that $d_a \approx d_{hor}(t_0)/(1+z)$ if z is large. A z of 1090 is large. These two results are numerically identical since they differ only by changing the lower limit on the dimensionless integral from $1/(1+z)$ to 0, which results in a negligible change. However, it is then the case that both equations require $\Omega_0 \approx 1.0$ in order to be accurate. But the changes in geometry encoded in $S_k(r/R_0)$ are significant if r/R_0 is not much smaller than 1. Now $r/R_0 = \sqrt{|\Omega_0 - 1|}I$, where I is the dimensionless integral and $I \approx 3.3$ for $z = 1090$. Thus, the requirement is $|\Omega_0 - 1| \ll 1/I^2 = 0.092$. For changes in the density parameter of 10%, this requirement is often not met. So these approximations to d_a are of insufficient accuracy for part 5 of the assignment.

In my program to evaluate d_a, I used $S_k(r) = r$ if $r/R_0 = \sqrt{|\Omega_0 - 1|}I < 10^{-3}$. This avoids numerical problems associated with R_0 becoming very large when Ω_0 is very close, but not exactly equal, to zero.

The angular size of the particle horizon at $z = 1090$ as seen by us is

$$\theta_{hor} = \frac{d_{hor}}{d_a}. \tag{12}$$

For the standard parameters of the universe, $d_a = 12.83$ Mpc and $\theta_{hor} = 1.157°$.

5. Changing each of the three density parameters by ±10% in turn produces the values for d_a in the fourth column of Table 1. The corresponding fractional change in d_a is

in the fifth column. The value of d_a changes very little when $\Omega_{r,0}$ is changed since the radiation density is much smaller than the matter and vacuum energy densities over almost the entire time interval between $z = 1090$ and now. The fractional change in d_a for changes in $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ are comparable, reflecting the recent change from matter domination to vacuum-energy domination. Simple explanations for the signs of the changes in d_a are more difficult to construct, since the change is due to a combination of the changing age, hence size, of the universe and the effects of changing geometry.

6. The value of θ_{hor} and its fractional change are in columns six and seven. There are significant changes in θ_{hor} for changes in all three density parameters, though changing $\Omega_{r,0}$ has the least effect.

7. The two lines near the end of Table 1 give the fractional changes in d_a and θ_{hor} for 10% changes in $\Omega_{\Lambda,0}$ while adjusting $\Omega_{m,0}$ so as to keep $\Omega_0 = 1$. The changes are smaller than for 10% changes in $\Omega_{\Lambda,0}$ and $\Omega_{m,0}$ alone. This smaller size suggests that θ_{hor} is primarily sensitive to changes in the geometry of the universe. Detailed calculations of the angular scale of the first peak of the power-spectrum of the fluctuations in the CMB, for which θ_{hor} is a first approximation, shows that it is insensitive to parameters other than Ω_0.

8. Replacing the cosmological constant by quintessence with $w = -0.5$ changes the dependence of the dark energy density on a to $\epsilon_Q = \epsilon_{Q,0} a^{-3(1+w)} = \epsilon_{Q,0} a^{-3/2}$ (Ryden equation 5.9). Then the equation for d_{hor} becomes

$$\begin{aligned} d_{hor}(z) &= \frac{c}{(1+z)H_0} \int_0^{1/(1+z)} \frac{da'}{\left(\Omega_{r,0} + \Omega_{m,0} a' + \Omega_{Q,0} (a')^{1-3w} + (1-\Omega_0)(a')^2\right)^{1/2}} \qquad (13) \\ &= \frac{c}{(1+z)H_0} \int_0^{1/(1+z)} \frac{da'}{\left(\Omega_{r,0} + \Omega_{m,0} a' + \Omega_{Q,0} (a')^{5/2} + (1-\Omega_0)(a')^2\right)^{1/2}} \qquad (14) \end{aligned}$$

Because the quintessence contributes negligibly to the energy density over the range of the integral, d_{hor}, is unchanged from the standard case. I confirmed this numerically, as is shown in the last line of Table 1.

The integral for the coordinate distance r that appears in the equations for d_a becomes

$$r = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{da'}{\left(\Omega_{r,0} + \Omega_{m,0} a' + \Omega_{Q,0} (a')^{5/2} + (1-\Omega_0)(a')^2\right)^{1/2}}. \qquad (15)$$

For $\Omega_{Q,0} = 0.73$, $d_a = 11.772$ Mpc and this is a fractional change of -0.082 from the standard value. The value of θ_{hor} is 1.262°, corresponding to a fractional change of $+0.090$. These values are in the last line of Table 1. This is among the larger changes calculated in this exercise. However, in practice it turns out that it is not easy to distinguish between a simple change in $\Omega_{\Lambda,0}$ and a change from a cosmological constant to quintessence using just the CMB.