Lecture 9. Charge Transport in Metals, Electron Scattering

- Quasi-classical approach, scattering time approximation
- Elastic Electron scattering
- Electron-phonon scattering
- Electron-electron scattering
\[ \sigma = \frac{e^2 n(T) \tau(T)}{m^*} = e n(T) \mu(T) \] - determined by the electron concentration \((n)\) and mobility \((\mu)\)

Semiconductors: \(n\) varies by many orders of magnitude with doping, temperature, etc.

Metals: \(n\) is fixed
Conductivity of Metals

Electron diffusion in a random scattering potential:

\[ \sigma = \frac{e^2 n(T) \tau(T)}{m^*} = en(T) \mu(T) \]

\( \tau \) – the momentum relaxation time

Degenerate Fermi gas (HW 3):

\[ \sigma_{3D} = e^2 g_{3D}(E_F) D_{3D} \quad (\Omega \cdot m)^{-1} \]

\[ D_{3D} = \frac{v_F l}{3} = \frac{v_F^2 \tau}{3} \]

\[ \sigma_{2D} = e^2 g_{2D}(E_F) D_{2D} \quad \Omega^{-1} \]

\[ D_{2D} = \frac{v_F l}{2} = \frac{v_F^2 \tau}{2} \]
Relaxation Time Approximation

\[ \vec{\tau} = -\frac{e^\tau}{\hbar} \vec{E} + \frac{eE\tau}{\hbar} \vec{v} \]

Assume that the free-electron picture is approx. valid, the electron states are close to the center of the BZ, \( \hbar k \approx p \), \( v_g \approx p/m^* \), Bloch oscillations are suppressed by frequent scattering.

Electron distribution in k-space when E-field is zero

Electron distribution is shifted in k-space when E-field is not zero

Definition:

- \( \vec{\tau} \equiv \tau_p \) – the momentum relaxation time
- \( \vec{v} \equiv \vec{v}_g \) - the group velocity (assume \( \varepsilon \propto k^2 \))

Typically, the FS shift is tiny:

\[ k \approx k_F \approx \frac{\pi}{a} \sim 10^{10} m^{-1} \]

\[ \frac{eE\tau}{\hbar} e \left( E = \frac{1}{m}, \tau \sim 10^{-14} s \text{ in metals at RT} \right) \sim 10 m^{-1} \]
Relaxation Time Approximation

Assume that the free-electron picture is approx. valid, the electron states are close to the center of the BZ, \( \hbar k \approx p \), \( v_g \approx p/m^* \), Bloch oscillations are suppressed by frequent scattering.

\[
\begin{align*}
\vec{j} &= -e \int 2f(\vec{k} + \frac{eE\tau}{\hbar}) \vec{v}(\vec{k}) \frac{dk_x dk_y}{(2\pi)^2} = -e \int f(\vec{k}) \vec{v} \left( \vec{k} - \frac{eE\tau}{\hbar} \right) \frac{dk_x dk_y}{2\pi^2} \\
&= -e \int f(\vec{k}) \hbar \left( \frac{\vec{k} - eE\tau}{\hbar} \right) \frac{dk_x dk_y}{m^* 2\pi^2} = -e \int f(\vec{k}) \frac{\hbar k}{m^*} \frac{dk_x dk_y}{2\pi^2} + \frac{e^2E\tau}{m^*} \int f(\vec{k}) \frac{dk_x dk_y}{2\pi^2} \\
&= \frac{e^2n\tau}{m^*} E = \sigma E \\
\sigma &= \frac{e^2n\tau_p}{m^*} \quad \text{- similar to Drude result, but the microscopic picture is different.} \\
\tau_p &- \text{the momentum relaxation time}
\end{align*}
\]
## Electron Scattering

<table>
<thead>
<tr>
<th>Elastic</th>
<th>Inelastic</th>
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</thead>
<tbody>
<tr>
<td>$(k \text{ changes, } \varepsilon = \text{ const})$</td>
<td>$(\text{both } k \text{ and } \varepsilon \text{ change})$</td>
</tr>
<tr>
<td>- Impurities</td>
<td>- Phonons</td>
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<tr>
<td>- Surfaces</td>
<td>- Other electrons</td>
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<tr>
<td>- Grain boundaries</td>
<td>- Other collective excitations</td>
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<tr>
<td>- Any deviations from translation symmetry</td>
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"Quenched Disorder"

<table>
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<tr>
<th>$\tau_p$</th>
<th>- momentum relaxation time</th>
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</thead>
<tbody>
<tr>
<td>$\tau_\varepsilon$</td>
<td>- energy relaxation time</td>
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$\tau_p$ and $\tau_\varepsilon$ can be orders of magnitude apart, even for the same scattering mechanism (e.g. $e$-$ph$ scattering at low $T$).
Impurities are randomly distributed, the scattering time is inversely proportional to the impurity density $n_i$ and the scattering cross-section $a_s^2$.

**Neutral impurities**: impenetrable spheres $a_s \sim r_B = \frac{\hbar^2}{me^2} \sim 10^{-10}m$ $a_s \sim 10^{-20}m^2$.

**Charged impurities**: in metals, also behave like impenetrable spheres due to *screening*.
Charged Impurities

Thomas–Fermi approximation of a slowly varying screened Coulomb potential $r_{TF} > \lambda_F$. $\delta \varphi$ redistributes the electron density near a charged impurity.

Poisson equation: $\nabla^2 (\delta \varphi (r)) = \frac{e^2}{\varepsilon_0} g_{3D}(E_F) \delta \varphi (r)$

Screened Coulomb potential, Thomas–Fermi approximation

$r_{TF} > \lambda_F$

\[
\varphi(r) = \frac{q}{4\pi\varepsilon_0 r} e^{-r/r_{TF}}
\]

\[
r_{TF} = \left( \frac{\varepsilon_0}{e^2} \frac{2E_F}{3n} \right)^{1/2} = \left( \frac{\varepsilon_0}{e^2 g_{3D}(E_F)} \right)^{1/2}
\]

$r_{TF}$ - Thomas–Fermi screening length

The assumption $r_{TF} > \lambda_F$ works well in semiconductors, but not in metals (estimate at home).
In metals, an impurity is surrounded by damped oscillations of the electron density, the period of oscillations $\sim \lambda_F$ (Friedel oscillations), $\lambda_F \sim a$ in metals.

Friedel oscillations can be visualized in the low-$T$ STM experiments. At high $T$ they are smeared (the range of energies around $E_F$ becomes sizable and the interference pattern disappears).
In metals, both neutral and charge impurities can be considered as impenetrable spheres with $a_s \approx 10^{-10} m$.

Effective scattering cross-section $a_s^2 \approx 10^{-20} m^{-2}$. In 1s, the electron travels over $v_F \cdot 1s$.

The time between collisions (the momentum relaxation time) is given by

$$\tau_p = \frac{1}{n_{imp} \cdot v_F \cdot a_s^2}$$

The mean free path is

$$l = \tau_p \cdot v_F = \frac{1}{n_{imp} \cdot a_s^2}$$

For the concentration of impurities 100ppm (very “clean” metal),

$$l \approx \frac{1}{10^{-4} a^{-3} a^2} = 10^4 a \approx 1 \text{ micron}$$

In this classical model the contribution to $\sigma$ due to impurity scattering in metals is $T$-independent (degenerate electron gas, $v_F \neq f(T)$) - the so-called residual resistivity.
Interference Effects?

Electrons are waves. Why can we ignore the interference effects?

Consider diffusive motion between points A and B. The electron can move along different trajectories. The total probability to get from A to B:

$$P_{AB} = \left| \sum_i A_i \right|^2 = \sum_i |A_i|^2 + \sum_{ij} |A_iA_j|$$

The phase gain for each trajectory:

$$\Delta \varphi = \frac{1}{\hbar} \int_A^B \vec{p}_F d\vec{l} \gg 1$$

The interference term vanishes after averaging for most of the trajectories

$$\left\langle \sum_{ij} |A_iA_j| \right\rangle = 0$$

As the result, the phase-related effects are washed away and the transport looks like a classical (“Sommerfeld-like”) diffusive transport.

However, quantum interference effects in $\sigma(T)$ can be observed at low $T$ – see L10.
If the surface roughness $\gtrsim \lambda_F$, the electron scattering is diffusive (in contrast to specular). This is almost always the case for metals

$$\frac{1}{l} = \frac{1}{l_{bulk}} + \alpha \frac{1}{t}$$

$t$ – film thickness

percolation threshold
Atomically-perfect interface, charged donors are removed from the conducting channel (only small-angle scattering): very long m.f.p. ($\sim 100 \mu m$).
Some Estimates

_Cu at T = 300K_

\[ n \approx 1 \cdot 10^{29} m^{-3} \quad m^* \approx m_e \approx 1 \cdot 10^{-30} kg \]

\[
\sigma = \frac{ne^2\tau}{m} = 2.6 \cdot 10^7 (\Omega \cdot m)^{-1}
\]

\[
\mu = \frac{e\tau}{m} \approx 1 \cdot 10^{-3} m^2/Vs
\]

\[
\begin{align*}
 l &= 1 \cdot 10^{-8} m \\
 \tau &= \frac{l}{v_F} = 1 \cdot 10^{-14} s
\end{align*}
\]

High-purity single crystals at low T

_Al single crystals with a residual resistance ratio \(\frac{R(300K)}{R(4.2K)} \approx 10^5 (!!!)\) at \(T = 4.2K\)


\[
l \approx 3 \cdot 10^{-3} m \quad \tau = \frac{l}{v_F} \approx \frac{3 \cdot 10^{-3} m}{1 \cdot 10^6 m/s} = 3 \cdot 10^{-9} s
\]

\[
\mu = \frac{e\tau}{m} \approx 500 m^2/Vs
\]

_Heterostructures at low T_

\[
m^* \approx 0.07 m_e \quad \tau \approx 1 ns
\]

Low-T mobility up to \(\mu \equiv \frac{e\tau}{m} \approx 3,000 \frac{m^2}{V \cdot s}\)

Low-T m. f. p. – up to \(\sim 10^{-3} m\)

- ballistic transport in small samples!
The Electron-Phonon Interaction depends on the phonon mode (acoustic or optical), polarization (transverse or longitudinal), and mechanism (deformation potential, polar, piezoelectric).

The phonon may be emitted or absorbed by an electron, causing the electron’s $p$ and $\epsilon$ to change. An electron can only emit a phonon with energy $\lesssim k_B T$, i.e. even though the electron distorts the lattice at low temperatures, it cannot emit energetic phonons because there are no accessible final electron states for such a process.

In thermal equilibrium between electrons and phonons, the probability to emit and absorb a phonon are the same, and both are proportional to the phonon DoS at the energy gained/lost by an electron.
Electron-Phonon Scattering at High $T \sim T_D$

$T \geq T_D$ — roughly speaking, all phonon modes are populated, the occupancy $\langle n(\omega) \rangle \propto T$, most phonons have energy $\sim k_BT_D$.

For both electrons with $\varepsilon = E_F$ and phonons with $\varepsilon \approx k_BT_D$, the values of $k$ are of the same order of magnitude $\sim \pi/a$, so electrons are scattered all over the Fermi surface.

The characteristic times of energy relaxation ($\tau_\varepsilon$) and momentum relaxation ($\tau_p$) are of the same order of magnitude.

At $T \sim T_D$, the probability of scattering $\propto T$ and $\tau_p \propto T^{-1}$

$\rho(T) \propto T$

$T_D(Pt) \approx 230K$

Platinum wire as a resistive “thermometer”
The intensity of $e - ph$ scattering depends on the relationship between the phonon wavelength $\lambda_{ph}(T)$ and the electron m.f.p. $l$.

**“Clean” case**

\[ \lambda_{ph}(T) \ll l \]

\[ \lambda_{ph}(T) = \frac{hc}{k_B T} \]

We’ll consider only the “clean” case.

**“Dirty” case**

\[ \lambda_{ph}(T) \gg l \]

$T = 1K, \ c = 3 \times 10^3 m/s, \ \lambda_{ph}(1K) = 0.14 \mu m$
Electron-Phonon Scattering at $T \ll T_D, \lambda_{ph}(T) \ll l$

Phonon energy $\sim k_B T$, the phonon momentum $k_{ph} \ll \frac{\pi}{a}$,

the 3D phonon $\text{DoS} \propto T^2$, 
the number of phonons $\propto T^3$ ($\text{DoS} \times \Delta \varepsilon$).

The probability of collision $\propto T^3$, and the energy relaxation time $\tau_\varepsilon \propto T^{-3}$.

Though the energy relaxation time $\tau_\varepsilon \propto T^{-3}$, the momentum relaxation time $\tau_p$ in this regime is much longer: the $e - ph$ scattering does not change much the direction of $k_e$ since $k_{ph}(\text{low } T) \ll \frac{\pi}{a} \sim k_e$ (small-angle scattering).

The inefficiency of collisions can be characterized by a weighting factor $1 - \cos \theta$, where $\theta$ is the scattering angle

$$1 - \cos \theta \approx \frac{\theta^2}{2} \approx \frac{k_{ph}^2}{2k_F^2} \approx \frac{\omega_{ph}^2}{2k_F^2 c^2} \propto T^2$$

and the momentum relaxation time $\tau_p \propto T^{-5}$.

$\rho(T) \propto T^5$
At intermediate $T$, there is an extrapolation function:

The following theoretical formula may be deduced for the electrical conductivity of a metal, subject to many simplifying assumptions concerning the interaction between the electrons and the lattice vibrations (see, for example, Wilson 1936, Chapter VI):

$$\frac{1}{\sigma_0} = \frac{\dot{m}\bar{v}}{n\varepsilon^2 l_r} + \left(\frac{m}{2}\right)^{1/2} \frac{9\pi\hbar^2 C^2}{8n\Delta e^2 Mk\Theta^2} \frac{(T)^5}{\Theta} \int_0^{\Theta/T} \frac{z^5 \, dz}{(e^z - 1)(1 - e^{-z})}.$$  \hspace{1cm} (2)

Here $k$ is Boltzmann's constant, $\zeta$ is the Fermi energy level ($\xi = \frac{1}{2}m\bar{v}^2$), $\Theta$ is the Debye temperature, $M$ is the mass of an atom, $\Delta$ is the volume of the unit cell, and $C$ is a constant which determines the interaction between the electrons and the lattice. According to this formula the 'ideal' and 'residual' resistances are additive, and the ideal resistance is proportional to $T$ at high and to $T^5$ at very low temperatures.
Can One Observe $\rho(T) \propto T^5$ in Experiment?

**Mattissen’s Rule**

An empirical rule which states that the total resistivity of a metallic sample is the sum of $\rho_{imp}$ due to impurity scattering and $\rho_{el-ph}$ due to phonon scattering (and other types of scattering if they act independently). Violations of the rule are observed as often as the rule itself...

$$\frac{1}{\tau} = \frac{1}{\tau_{imp}} + \frac{1}{\tau_{el-ph}}$$

$$\rho = \rho_{imp} + \rho_{el-ph} = \frac{m}{ne^2} \left( \frac{1}{\tau_{imp}} + \frac{1}{\tau_{el-ph}} \right)$$
Can One Observe $\rho(T) \propto T^5$ in Experiment? (cont’d)

$T^5$ can be observed only at $T < \frac{T_D}{10}$ in ultra-clean samples - $RRR > 10^3$
Implies $l_{imp} > 10\mu m$ (!)

 ultra-clean single crystals
Violation of Mattissen Rule: inelastic scattering by vibrating impurities

(Elastic) contribution to the resistivity due to e-ph-imp interference (mainly due to interaction with transverse phonons)

\[
\frac{\Delta \rho_{e-ph-imp}}{\rho_0} = \left[ 2 \left( \frac{u_l}{u_i} \right)^3 + \frac{\pi^2}{16} - 1 \right] \frac{2 \pi^2 \beta}{3 \varepsilon_F p_F u_i} (k_B T)^2
\]

Reizer and Sergeev, Sov. Phys. JETP 65, 1291 (1987)

\[
\Delta \rho_{el-ph-imp} + \Delta \rho_{BG} \propto T^{-5}
\]
\[
\Delta \rho_{el-ph-imp} \propto T^{-2}
\]

at \( T < T_D/10 \)


Au film \( t \sim l \sim 30 \text{nm} \)
Electron-Electron Scattering

Consider the free-electron model.

Energy and momentum conservation:
\[ \epsilon(p_1) + \epsilon(p_2) = \epsilon(p_1') + \epsilon(p_2') \]
\[ \vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2' \]

\[ p_1, p_1', p_2' > p_F, \quad p_2 < p_F \]

Scattering probability: (all \( \epsilon \approx E_F \))
\[ Pr \propto \int \delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') dp_2 dp_1' \propto (p_1 - p_F)^2 \]
\( (p_2' \text{ is fixed by momentum conservation}) \)

\[ \epsilon - E_F = v_F (p - p_F), \text{ and } Pr \propto (\epsilon - E_F)^2 \]

Dimensionality approach:
\[ \tau_{e-e} \sim \frac{\hbar E_F}{(\epsilon - E_F)^2} \quad \text{all energies, both kinetic and potential, are } \sim E_F \]

\[ \tau_{e-e} \sim \frac{\hbar}{k_B T} \frac{E_F}{k_B T} \]
- the lifetime of an electron quasiparticle of energy \( \sim T \) (relative to the Fermi energy) in the Landau theory of Fermi liquids.

Typical \( \tau_{e-e} (1K) \approx 10^{-6} - 10^{-7} \text{s} \).
Typical energy of electron-electron interactions: $U \sim e^2 / a$.

For typical metals, $U \sim E_F$ - electrons strongly interact with each other.

**Landau Theory of Fermi Liquid:** Elementary excitations (quasiparticles) in a system of interacting electrons have properties similar to elementary excitations of a free electron gas. These elementary excitations in solids involve many particles and have a *collective character*.

\[
\begin{align*}
\text{"Electrons"}: \quad & \varepsilon_e = \frac{p^2}{2m} - \frac{p_F^2}{2m} \\
\text{"Holes"}: \quad & \varepsilon_h = \frac{p_F^2}{2m} - \frac{p^2}{2m}
\end{align*}
\]

Wave function of a quasiparticle:

\[
\psi \propto \exp\left(i \frac{\varepsilon}{\hbar} t\right) \exp(-\gamma t)
\]

$\gamma$ — damping due to electron-electron interactions.

We can treat excitations as “particles” that are characterized by their energy $\varepsilon$ and momentum $p$ only if their lifetime, $1/\gamma$, is sufficiently long: $\gamma \ll \varepsilon / \hbar$.

\[
\gamma = \frac{1}{\tau_{e-e}} \propto \frac{k_B T}{\hbar} \frac{k_B T}{E_F} \approx \frac{\varepsilon}{\hbar} \frac{k_B T}{E_F} \ll \frac{\varepsilon}{\hbar}
\]

$\varepsilon$ is the *qp* energy relative to the Fermi energy.

**Important conclusion:** electron excitations near the Fermi level in metals can be treated as quasiparticles provided $\varepsilon \ll E_F$. For typical metals, this condition is always met.
The Effect of Electron-Electron Scattering on Conductivity

Can we plug $\tau_{e-e}$ into the Drude formula? – No! If the total momentum of the electron gas is conserved, no effect of conductivity.

However, there are processes that do not conserve the quasi-momentum - the Umklapp scattering.

The Umklapp contribution to the conductivity, $\rho_U$, becomes significant if the Fermi surface is close to the BZ boundary. However, it rarely competes with impurities and phonons. Notable exceptions:

- Very clean metals at low $T$ (at high $T$, the $e-\text{ph}$ scattering dominates).
- Very large DoS at the Fermi energy - this bumps up the number of initial and final states (e.g. transition metals, heavy fermion compounds).

$$\rho_U \propto \tau_{e-e}^{-1} \propto T^2$$

$\rho(T)$ of single crystal Yb$_2$Ni$_{12}$P$_7$ at several applied pressures up to 2.4 GPa.

Charge transport is controlled by different scattering processes: $e-e$, $e-ph$, $e-...$

Elastic and Inelastic scattering processes: the momentum and energy relaxation times can be many orders of magnitude apart.

Electron-phonon scattering in clean case, Bloch-Grüneisen Law

Electron-electron scattering, justification of the quasiparticle picture.

Applications: importance of understanding the energy relaxation, hot-electron effects.