Lecture 20. Josephson Effects

- Quasiparticle and Cooper pair tunneling between superconductors
- The DC Josephson effect
- The AC Josephson effect
- \( I - V \) characteristics of Josephson Junctions
- Applications of Josephson effects
Superconductivity: Macroscopic Quantum State

Cooper Pairs: two bound electrons with opposite momenta and spins (*bosons*).

Ground state $\Rightarrow$ superfluid pair condensate

CP size $\xi_0 \sim 0.1 - 1 \mu m$

All Cooper pairs are condensed into a single state described by a macroscopic many-particle condensate wavefunction:

$$\Psi(r, t) = |\Psi(r, t)| e^{i\varphi(r,t)} = \sqrt{n_S(r, t)} e^{i\varphi(r,t)}$$

- **CP density**
- **macroscopic phase**
Quasiparticle Tunneling between Two Superconductors

This is tunneling of quasiparticles. Can Cooper pairs tunnel as well?

This is not obvious – if one (erroneously) considers CP tunneling as uncorrelated tunneling of two quasiparticles, the probability of tunneling would be too small.

What I missed……

Now known as the Josephson Effect
Cooper Pair Tunneling: The Josephson Effects

In each electrode all Cooper pairs are condensed into a single state described by a macroscopic many-particle condensate wavefunction:

$$\Psi(r, t) = \sqrt{n_S(r, t)} \ e^{i \varphi(r, t)}$$

CP density macroscopic phase

What happens if we consider tunneling of the macroscopic wave function?

Simplified derivation of Josephson effects (Feynman): treat the JJ as a two-level system, where a CP can be either in state 1 (left electrode) or in state 2 (right electrode). The corresponding CP energies are $eV$ and $-eV$. The wavefunctions are $\Psi_1 = \sqrt{n_S} \ e^{i \varphi_1}$ and $\Psi_2 = \sqrt{n_S} \ e^{i \varphi_2}$.

$$i \hbar \frac{d \Psi_1}{dt} = eV \Psi_1 + K \Psi_2$$
$$i \hbar \frac{d \Psi_2}{dt} = -eV \Psi_2 + K \Psi_1$$

$K$ - the “transparency” of the junction. If $K = 0$, these Eqs describe the lowest energy state of both electrodes.
After separating the real and imaginary parts:

\[ \frac{dn_s}{dt} = \frac{2Kn_s}{\hbar} \sin(\varphi_2 - \varphi_1) = \frac{2Kn_s}{\hbar} \sin \varphi \]

\[ \frac{d\varphi_1}{dt} = -\frac{K}{\hbar} \cos \varphi - \frac{eV}{\hbar} \]

\[ \frac{d\varphi_2}{dt} = -\frac{K}{\hbar} \cos \varphi + \frac{eV}{\hbar} \]

\[ \frac{d(\varphi_2 - \varphi_1)}{dt} = \frac{d\varphi}{dt} = \frac{2eV}{\hbar} \]

The supercurrent flows whenever there is a difference \( \varphi_2 - \varphi_1 \).

Vice versa, the supercurrent flow through the JJ generates the phase difference \( \varphi_2 - \varphi_1 \).
The DC Josephson Effect

\[ V = 0 \quad \frac{d\varphi}{dt} = \frac{2eV}{\hbar} = 0 \quad \varphi_2 - \varphi_1 \neq f(t) \]

Current of Cooper pairs:
\[ \frac{dn_s}{dt} = \frac{2Kn_s}{\hbar} \sin \varphi \]

**the DC J. effect**

\[ I_S(\varphi) = I_C \sin \varphi \]

\[ I_C(T = 0) = \frac{\pi \Delta(0)}{2eR_n} \]

- microscopic theory

\[ I_S \] - the supercurrent (CP flows at zero voltage difference)

\[ I_C \] – max current that can flow through a junction without dissipation

\[ R_n \] - the resistance of the tunnel junction at \( T > T_C \), is inversely proportional to the probability of tunneling of quasiparticles through the tunnel barrier.

\[ \Delta(0) \] - the energy gap at \( T = 0 \)

The supercurrent (no dissipation, \( V = 0 \)) can flow between the electrodes due to the tunneling of Cooper pairs.

Cooper pairs can tunnel coherently through a tunnel barrier between two superconductors with the same probability as single quasiparticles.
D.G. McDonald, Physics Today, July 2001, p.46
The DC Josephson Effect and the Josephson Energy

At non-zero $T$:

$$I_C(T) = \frac{\pi \Delta}{2eR_n} \tanh \left( \frac{\Delta}{2k_B T} \right)$$

Josephson energy:

$$E_J(\varphi) = \frac{\hbar I_C}{2e} (1 - \cos \varphi) = \frac{\Phi_0 I_C}{2\pi} (1 - \cos \varphi)$$

By this amount the total free energy of two superconducting electrodes is decreased due to the tunneling coupling between them.

$$E_{J0} \equiv \frac{\hbar I_C}{2e} = \frac{\pi \hbar \Delta}{(2e)^2 R_N} = \pi \Delta \frac{R_{Qc}}{R_N}$$

$$R_{Qc} \equiv \frac{\hbar}{(2e)^2} = \frac{R_Q}{8\pi} \approx 1\,k\Omega$$

$$R_Q \equiv \frac{\hbar}{e^2} \approx 25.8\,k\Omega$$
The AC (non-stationary) Josephson Effect

What happens if the current exceeds $I_C$? The junction cannot support a dissipationless current, there is a voltage drop between two superconductors, and the phase difference becomes time-dependent.

$$\frac{d\phi}{dt} = \frac{2eV}{\hbar}$$

The AC Josephson effect is used in metrology: conversion of frequency into voltage. However, the **non-zero $V$ is not acceptable for the qubits** (dissipation, radiation, etc.).

The resistively shunted Josephson junction (RSJ) model

$C$ – the capacitance of the parallel-plate capacitor formed by the electrodes.

$R$ includes all possible losses, including the $qp$ dissipation at $T > 0$, the e.m. radiation from the junction, etc.
If $R = \infty$ - the underdamped regime ($T \ll T_C$, no external resistive shunt):

\[
\frac{dQ}{dt} + I_C \sin \varphi = I \quad \frac{\hbar C \, d^2 \varphi}{2e \, dt^2} + I_C \sin \varphi = I
\]

**Josephson plasma frequency**

\[
\omega_p = \sqrt{\frac{2eI_C}{\hbar C}} = \frac{1}{\sqrt{L_j C}}
\]

- $L_j$ - the Josephson inductance (see below)

small oscillations at $\omega_p$

Motion of a “particle” with coordinate $\varphi$ and mass $\propto C$ in a “washboard” potential:

\[
U(\varphi) = E_J \left\{ (1 - \cos \varphi) - \frac{I}{I_C} \varphi \right\}
\]
The I-V curves are strongly hysteretic – when the “washboard” potential is returned to its horizontal position (small $I$), the “particle” still has sufficient kinetic energy to overcome potential maxima.
The $I-V$ curves are non-hysteretic: as soon as $I < I_c$, the phase will be re-trapped at one of the potential minima due to strong dissipation.

If $C = 0$:

$$\frac{V}{R} + I_c \sin \varphi = I \quad \frac{\hbar}{2eR} \frac{d\varphi(t)}{dt} + I_c \sin[\varphi(t)] = I$$

$$V(t) = R \frac{I^2 - I_c^2}{I + I_c \cos(\omega t)}$$

$$\omega = \frac{2e}{\hbar} R \sqrt{I^2 - I_c^2}$$

What is $V_{\text{max}} - V_{\text{min}}$?

$$\langle V \rangle = R \sqrt{I^2 - I_c^2}$$

$\langle V \rangle$ - the time-averaged voltage difference across the junction
When the “particle” (phase) starts “sliding down”, its motion depends on the **energy dissipation** in the junction (“damping”, or $R$).

The McCumber-Stewart parameter

$$\beta_c = \frac{2eI_cR^2C}{\hbar} \ll 1$$

**Overdamped junction**
- Capacitance & resistance small
- $M$ small, $\eta$ large
- Non-hysteretic IVC

(Phase particle will retract immediately at $I_c$ because of large damping)

$$\beta_c = \frac{2eI_cR^2C}{\hbar} \gg 1$$

**Underdamped junction**
- Capacitance & resistance large
- $M$ large, $\eta$ small
- Hysteretic IVC

(Once the phase is moving, the potential has to be tilted back almost into the horizontal position to stop its motion)

http://www.wmi.badw-muenchen.de/teaching/Lecturenotes/AS/AS2015_Chapter03_Slides.pdf
Overdamped Junctions

Time-averaged currents and voltages

\[ \bar{i} = \frac{\bar{I}}{I_C} \]

\[ \bar{V} = \bar{\phi} = \frac{V}{V_C} \]
• for $I \geq I_c$:
  highly non-sinusoidal oscillations
• long oscillation period
• $\langle V \rangle \propto 1/T$: small

• for $I \gg I_c$:
  almost all current flows as normal current
• junction voltage is about constant
• oscillations of Josephson current are almost sinusoidal
• time averaged Josephson current almost zero
• linear/Ohmic IVC

→ analogy to pendulum
The Josephson Frequency

\[ V(t) = \frac{\hbar}{2e} \frac{d\varphi}{dt} \quad \int V(t)dt = \frac{\hbar}{2e} 2\pi = \Phi_0 \]

**the Josephson frequency:**

\[ f_j = \frac{2e}{h} \langle V \rangle \approx 0.5 \frac{GHz}{\mu V} \langle V \rangle \]

Here \( V \) is the time-averaged value of the voltage difference across the JJ.

Periodic variations of voltage at the primary frequency \( f_j \) occur if the junction is biased with \( I > I_c \). The generation of voltage spikes can also be described as a result of the flux quanta transfer across the Josephson junction.
The Josephson junctions driven by external microwaves with frequency $f$ become phase-locked with the radiation, their $I-V$ curves demonstrate the constant-voltage steps at $V_n = n(h/2e)f$ ( $n = 1, 2, 3, ...$). These so-called Shapiro steps are used for the realization of voltage standards.

Many Josephson junctions connected in series – greater $V$, up to 10V. Many Josephson junctions connected in parallel – greater $I$. 

$$f_j = \frac{2e}{h} V \approx 0.5 \frac{GHz}{\mu V}$$
The macroscopic coherence of Cooper pairs $\implies$ periodic response of multi-connected superconductors to external magnetic field.

$$\gamma = \varphi - \frac{2\pi}{\Phi_0} \int \vec{B} \cdot d\vec{s}$$

$$I = I_C (\sin \gamma_1 - \sin \gamma_2) = I_C \left[ \sin \gamma_1 + \sin \left( \gamma_1 - \frac{2\pi \Phi}{\Phi_0} \right) \right] = 2I_C \cos \frac{\pi \Phi}{\Phi_0} \sin \left( \gamma_1 - \frac{\pi \Phi}{\Phi_0} \right)$$

$$I_{\max} = 2I_C \left| \cos \left( \pi \frac{\Phi}{\Phi_0} \right) \right|$$

- two-slit experiment with Cooper pairs

Symmetric DC SQUID: Josephson element with $B$-tuneable Josephson energy.

At $\Phi = \frac{1}{2} \Phi_0$, $E_j^{eff} = 0$, and coupling between two S.C. electrodes vanishes.
Aharonov-Bohm Effect

A particle with electric charge $q$ travelling along some path in a region with zero magnetic field, but a non-zero vector-potential $A$, acquires a phase shift

$$\delta \varphi = \frac{q}{\hbar} \int_A \cdot d\vec{l}$$

The DC SQUID provides an analogue of the double-slit experiment, where the relative phase between the amplitudes of two processes – Cooper pair tunneling along the upper and lower paths – is controlled by the magnetic flux in the loop.

$$\Delta \varphi = \frac{2e}{\hbar} \int_A \cdot d\vec{l} = \frac{2e}{\hbar} \Phi = 2\pi \frac{\Phi}{\Phi_0}$$

At $\Phi = n\Phi_0$ - constructive interference, greatest probability of tunneling, and largest $E_f$.

At $\Phi = (n + 1/2)\Phi_0$ - destructive interference, zero probability of tunneling, and $E_f = 0$. 

Overdamped regime (junctions are shunted by external $R$).

Ideal DC SQUID ($L$ of the loop can be neglected).

Voltage modulation by $\Phi$ at a fixed $I_B$.

Lowest flux noise $S_\Phi \approx 10^{-6}\Phi_0/\sqrt{Hz}$ - resolution of $B$ measurement in 1 s $\sim 10^{-13} G$
The Josephson Inductance

Let’s consider harmonic approximation $|\varphi| \ll \pi$

\[
\begin{align*}
I &= I_c \varphi \\
\dot{I} &= I_c \dot{\varphi} = I_c \frac{2eV}{\hbar} \\
L_j &\equiv \frac{V}{I} = \frac{\hbar}{2eI_c} \quad \text{-- the JJ can be viewed as an inductor!}
\end{align*}
\]

The nature of this inductance – the inertia of Cooper pairs. The energy is stored in the CP kinetic energy rather than in the magnetic field!

\[
\begin{align*}
I &= I_c \sin \varphi \\
\delta I &= I_c \cos \varphi \delta \varphi \\
\delta V &= \frac{\hbar}{2e} \frac{d(\delta \varphi)}{dt} = \frac{\hbar}{2eI_c \cos \varphi} \frac{1}{dt} \frac{d(\delta I)}{dt} = L \frac{d(\delta I)}{dt}
\end{align*}
\]

\[
L = \frac{\hbar}{2eI_c \cos \varphi} = \frac{L_j}{\cos \varphi}
\]

The Josephson element \(\Rightarrow\) a non-linear inductor!

Josephson inductance – the inertia of Cooper pairs.

\(I_c = 100nA \rightarrow 0.1\mu H\), the inductance as of a 10-cm-long wire!
Josephson Junction as a Non-Linear Oscillator

The Josephson junction is a non-dissipative (at $T \rightarrow 0$) nonlinear inductor shunted by a capacitor

$\Rightarrow \text{a non-linear non-dissipative oscillator.}$

**Josephson energy**

$$E_J = \frac{\hbar I_c}{2e} = \left(\frac{\Phi_0}{2\pi}\right)^2 \frac{1}{L_J}$$

Mag. flux quantum $\Phi_0 = \frac{h}{2e} \approx 20G \cdot \mu m^2$

**Charging energy**

$$E_C = \frac{(2e)^2}{2C_J}$$

**Josephson plasma frequency**

$$\omega_p = \frac{1}{\sqrt{L_J C}} = \frac{\sqrt{2E_J E_C}}{\hbar}$$

- the Josephson plasma frequency (typically $\sim 100GHz \sim 5K$)

- depends on oxide transparency, not on the JJ area

**Josephson junction impedance:**

$$Z_J = \sqrt{\frac{L_J}{C}} \approx 1k\Omega \sqrt{\frac{2E_C}{E_J}}$$

- tunable, $E_j/E_C \sim (\text{JJ area})^2$
Estimates for Small Junctions

\[ E_J = \frac{\pi \hbar \Delta(0)}{(2e)^2 R_n} \]

The resistance of the tunnel junction in the normal state \( R_n = 1 \, k\Omega \)

\[ E_J \approx 7K \]

\[ E_C = \frac{(2e)^2}{2C_j} \]

The width of each electrode \( W = 0.15\mu m \)

\[ C_j \approx 50 \, fF/\mu m^2 \]

\[ E_C \approx 3K \]

typical

Micro energy controls a macro system!

Small Josephson junction \( \Rightarrow \) a “macroscopic nucleus with wires”

[Clarke et al., *Science* **239**, 992 (’88)]

Josephson junctions, being non-dissipative and nonlinear oscillators offer a platform for implementation of superconducting quantum bits.
Conclusions

- DC and AC Josephson effects: quantum tunneling of Cooper pairs between superconductors.

- Applications:
  - SQUIDs (detectors of weak magnetic fields);
  - Digital superconducting electronics (low energy dissipation, $f$ up to 100 GHz);
  - Josephson voltage standards;
  - Amplifiers/generators of microwave radiation;
  - Superconducting quantum bits;
  - many more.