Lecture 14. Integer Quantum Hall Effect

- Ideal \((\omega_C \tau \to \infty)\) conductor in magnetic field
- IQH effect: experimental facts
- Edge channels
- Explanation: Landau levels + localization + charge quantization

“Grand unification”: disorder-induced localization, quantum channels, Landau levels, Aharonov-Bohm phase...

Probably, the most counter-intuitive lecture...
Infinite-in-\(xy\)-plane Ideal (\(\omega_C \tau \rightarrow \infty\)) 2D Conductor in \(B\)

\[
\rho = \begin{pmatrix}
\frac{m}{e^2 n \tau} & -\frac{B}{en} \\
\frac{B}{en} & \frac{m}{e^2 n \tau}
\end{pmatrix}
\]

For an infinite ideal 2D system (no disorder) \(\tau \rightarrow \infty\):

\[
\rho = \begin{pmatrix}
0 & \frac{B}{ne} \\
-\frac{B}{ne} & 0
\end{pmatrix}
\]

\[
\sigma = \frac{ne^2 \tau}{m} \begin{pmatrix}
1 & \omega_C \tau \\
-\omega_C \tau & 1
\end{pmatrix} \approx \frac{ne^2}{\omega_C^2 \tau} \begin{pmatrix}
1 & \omega_C \tau \\
-\omega_C \tau & 1
\end{pmatrix} = \begin{pmatrix}
\frac{ne^2}{\omega_C^2 m \tau} & \frac{ne^2}{\omega_C m} \\
-\frac{ne^2}{\omega_C m} & \frac{ne^2}{\omega_C^2 m \tau}
\end{pmatrix}
\]

\[
\sigma_{xy} = \frac{ne^2}{\omega_C m} = \frac{ne}{B}
\]

\[
\sigma = \begin{pmatrix}
0 & \frac{ne}{B} \\
-\frac{ne}{B} & 0
\end{pmatrix}
\]

\[
\sigma_{xx} = \rho_{xx} = 0
\]

Both \(\sigma_{xx}\) and \(\rho_{xx}\) vanish at the same time! Curious situation - on one hand, \(\sigma_L = 0\) (“perfect insulator”), on the other hand, \(\rho_L = 0\) (“perfect conductor”).

\[
E_x = \rho_{xy} j_y
\]

\[
E_y = \rho_{yx} j_x
\]

\(2r_L\)
What happens if the electric field is applied? \( \sigma_{xx} = \rho_{xx} = 0 \)

In a disorder-free system, localization in the “bulk” is “soft” – weak \( E_x \) will induce current in the \( \gamma \)-direction.

\[
E_x = \rho_{xy} j_y \quad E_y = \rho_{yx} j_x
\]

In this situation \( \vec{E} \cdot \vec{j} = 0 \), so a non-dissipative current flows along equipotentials rather than along the \( E \) field lines.

**Role of disorder:** Disorder scatters electrons from one orbit to another, and delocalizes electrons. Recall Anderson Localization: inelastic scattering played the same role.

Delocalization is significant when \( l \lesssim r_L \quad (\omega_C \tau \lesssim 1) \)

Diffusion of an electron in the plane perpendicular to the magnetic field. Thick line shows the random diffusion path.
Consider $E = 0$. In a sample with no disorder, there are localized states in the “bulk” and delocalized motion along the edges due to *scattering*. Electrons travel along the full extent of the edges following *skipping orbits*. On each edge there are only *states moving in one direction*, the direction is determined by the relative orientation of $\vec{B}$ and the edge itself. Each edge carries a current, *even if there is no E field*.

A particle restricted to move in a single direction along a line is said to be chiral. The direction is opposite for opposite edges of the sample (the electrons have opposite chirality on the two edges). This ensures that the net current, in the absence of an electric field, vanishes.
The Hall bar - the sample of choice for most QH experiments. In the experiment, we control current $I_{12} (= I_{xx}$ far from the contacts $1,2), I_{xy} = 0$. Both $V_{xy}$ and $V_{xx}$ are measured in the 4-probe configuration.

**Non-uniform current distribution**

In the limit $\omega_C \tau = \infty$, $\vec{E} \cdot \vec{j} = 0$ and the current flows along equipotentials rather than along E field lines. Far from the current contacts, $\vec{j}$ must flow along the length of the sample, and $\vec{E}$ is directed across the sample, giving $V_H$ ($E_{xx} = 0$). This pattern breaks near the contacts since they must be equipotential. Near the contacts, $\vec{E}$ and $\vec{j}$ remain mutually orthogonal, all currents enter and leave at two opposite corners of the sample, the only places where the potential difference can occur along the sample. **All the power is dissipated near these two points** (everywhere else the power $\vec{E} \cdot \vec{j} = 0$ and current flows without dissipation).

To avoid messy situation near the current contacts, the voltage probes are attached far from the these contacts.
Weak $B$, $\omega_C \tau \ll 1$

Strong $B$, $\omega_C \tau \gtrsim 1$

2D Systems, Landau Levels

\[ E_{\perp,i} = \left( i + \frac{1}{2} \right) \hbar \omega_C \]

- Landau levels, \( i = 0, 1, 2, \ldots \)

with disorder

\[ \hbar \omega_C \leftrightarrow \hbar / \tau \]

\[ \frac{\hbar \omega_C}{\hbar / \tau} = \omega_C \tau = \frac{l}{r_L} = \mu B \]

ideal 2D system, no disorder

\[ g_{2D}(E) \]

\( \omega_C \tau \ll 1 \) – weak \( B \)/ strong disorder, “smeared” Landau levels

\( \omega_C \tau \gtrsim 1 \) – strong \( B \)/ weak disorder, well-resolved Landau levels
If $g = 2$, the states with $(i, s = +\frac{1}{2})$ and $(i + 1, s = -\frac{1}{2})$ are degenerate, and the spin-related degeneracy is absent only for the lowest LL. The degeneracy is the same if $g = 0$ (except for the lowest level).

The number of filled Landau levels = the filling factor

$$n_L = \hbar \omega_c \times g_{2D}(\epsilon) = \hbar \frac{eB}{m} \times \frac{m}{\pi \hbar^2} = \frac{2e}{\hbar} B = \frac{B}{\Phi_0} \left[ \# \right]$$

$$\nu = \frac{n_{2D}}{B} \Phi_0$$

Of course, $\nu$ can be non-integer: e.g., $\nu = 3.2$ means that three lowest L. levels are fully filled, and the fourth L. level is partially occupied.
In general, \( g \neq 2 \) due to the spin-orbit interaction in solids (e.g. \( g = -0.4 \) in bulk GaAs). As a result, there is an additional splitting of Landau levels, and each level can accommodate the following number of electrons:

\[
n_{L}/2 = \frac{e}{\hbar} B = \frac{B}{2\Phi_0} \left[ \frac{\#}{m^2} \right] \quad \text{ (each L. level is “spin-polarized”)}
\]

The filling factor

\[
v = \frac{n_{2D}}{B} 2\Phi_0
\]

For all problems (HW + exams) we will take spin splitting into account and use

Consider a 2D system of electrons with the electron concentration \( n_{2D} = 3.6 \times 10^{15} m^{-2} \). Estimate the smallest perpendicular magnetic field at which only one Landau level is filled \( (v = 1) \). Take into account spin splitting of Landau levels.

\[
v = \frac{n_{2D}}{B} 2\Phi_0 = 1 \quad B = n_{2D} 2\Phi_0 \approx 3.5 \times 10^{15} m^{-2} \times 2 \times 2 \times 10^{-15} T \cdot m^2 = 14T
\]

At this \( B \) in GaAs \( (m^* \approx 0.07 m_e) \)

\[
\hbar \omega_c = \hbar \frac{eB}{m^*} \approx 270K
\]
Integer QH regime:

\[ R_{xy} = \frac{\hbar}{ie^2} \]

\[ i = \{1, 2, 3, \ldots\} \]

Classical Hall effect regime:

\[ \Delta \rho_{xx}(B) = 0 \]

\[ \rho_{xy} = \frac{B}{|e|n} \]

Shubnikov-de Haas regime:

\[ \frac{\Delta \rho_{xx}(B)}{\rho} \ll 1 \]

\[ \rho_{xy} = \frac{B}{|e|n} \]
Requirements for the QHE Observation

- high-mobility samples (scattering time $\gg$ the inverse of the cyclotron frequency, or well-resolved L. levels):
  $$\omega_C \tau \gtrsim 1 \; \text{or} \; \frac{l}{r_L} \gtrsim 1 \; \text{or} \; \mu B \gtrsim 1$$

- sufficiently low $T$ (no electrons thermally excited to higher L. levels):
  $$k_B T \ll \hbar \omega_C$$

- just a few filled Landau levels: the filling factor $\nu \equiv \frac{n_{2D}}{B} \lesssim \text{few}$
Fixed \( n \), changing \( B \)

\[ \rho_{xx} \text{ becomes } \sim 0 \text{ within some intervals of } B. \]

However, the most surprising fact is **precise quantization of \( \rho_{xy} \) in units of** \( \frac{\hbar}{ie^2} \) (precision – better than one part in \( 10^9 \) !). This quantization is not affected by (moderate) disorder, el.-el. interactions, shape of the sample and how clean it is.

Fixed \( B \), changing \( n \)

\( \rho_{xy} \) quantization: quantization of a **macroscopic** property in a **disordered** system involving **many** particles. Its explanation requires something new!

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J. Weis, and K. von Klitzing  Phil. Trans. R. Soc. A 369:3954 (2011)
Though the quantization of $\rho_{xy}$ is very precise, other features vary from sample to sample.

**With increasing $T$:** the QH plateaus shrink, and $\rho_{xx}$ increases exponentially $\propto e^{-\Delta/k_B T}$. The activation energy $\Delta$ is the greatest in the middle of the QH plateaus.
Incorrect attempt to understand the phenomenon

Fixed $n$, changing $B$

Indeed, $n_{2D} = v n_i = v \frac{eB}{h}$.

It is easy to argue that the Hall resistivity should take quantized values $\rho_{xy} = \frac{h}{ie^2}$ when $\nu$ Landau levels are completely filled.

Using $\rho_{xy} = \frac{B}{ne}$ we obtain $\rho_{xy} = \frac{h}{ve^2}$ (lifted spin degeneracy)

However, this holds only for a single $B$ value (if $n_{2D}$ is fixed) or $n_{2D}$ (if $B$ is fixed). Indeed, in a perfectly clean system ($\delta$-function LLs), the Fermi energy “hops” from level to level. Thus, this doesn’t explain the emergence of $\rho_{xy}$ plateaus, with the quantization persisting over a range of magnetic fields.

Two observations to explain: (a) $\rho_{xy}$ plateaus and (b) the precision of quantization.
Importance of Disorder

Short-Range ($L < r_L$) Disorder (not very relevant to experiments)

Scattering leads to a broadening of Landau levels. The states within the low-DoS energy intervals are localized (recall band tails), whereas the states close to the centers of LLs are delocalized. **Now $E_F$ can be in the mobility (Landau) gap!** Both short- and long-range disorder lead to the “hard” electron localization (in contrast to “soft” localization in disorder-free samples) – application of electric field cannot delocalize the trapped electron states.

Since the localized states cannot carry current, $\rho_{xy}$ doesn’t change as long as $E_F$ remains in the mobility gap. When $E_F$ moves through the region of extended (delocalized) states, $\rho_{xy}$ makes a transition from one plateau to the next, and a non-zero $\rho_{xx}$ appears.

Of course, the arguments hold if the random potential is not too strong ($\frac{1}{t} \ll \omega_C$ or $\mu B \gg 1$), otherwise it is no longer useful to employ the Landau levels as starting points. Thus, **disorder is essential**, but it shouldn’t be too strong.

**The amazing precision of $\rho_{xy}$ quantization is due to the presence of disorder in the sample!**
Since the QH experiments are typically done on very clean samples, the long-range disorder is more important than the short-range one.

In magnetic field, the electrons follow the trajectories that correspond to iso-energy contours within the energy landscape. For a large-size sample, virtually all states are trapped by local potential fluctuations that are either above or below the mean value.

When the first LL is at least 50% filled, there will be a percolation path between the edges of the "lakes" and "mountains" and current will be able to pass through the bulk of the Hall bar. **If the Fermi level is in the mobility gap, the current can only pass from one contact to the next along the edge of the sample.**

The existence of localized states allows the Fermi energy to be positioned between the LLs, but it doesn’t affect the Hall conductivity (they act as “holes” in the sample, but punching holes doesn’t change $\rho_{xy}$ - no difference between “Emmentaler” and “Gruyère” geometries ).
No spin-splitting in this simulation.
In the Hall bars there are two edges without potential contacts. Near the edge of the sample the energy of electrons on the $i^{\text{th}}$ LL increases.

Using classical analogy, we can think of an increase of $\omega_C$ ("half-of-the-circular-orbit"), or quantum confinement.

The energy of an orbit on $i^{\text{th}}$ LL with the center at the boundary equals to the energy of an orbit in the bulk at the $(2i + \frac{3}{2})$ LL. The LLs near the boundary can carry current even if in the bulk $E_F$ is in the mobility gap. Thus, the edges are never gapped (provided there is at least one filled LL) whereas the bulk can be gapped or not depending on the position of $E_F$. 

$$r_i = \sqrt{2 \frac{\hbar}{e B} \left( i + \frac{1}{2} \right)}$$
For each edge, **there are as many edge states as there are filled Landau levels in the bulk of the system.**

The edge currents have a number of remarkable properties. Current cannot leak away from an edge because the bulk is gapped. Moreover, the chirality of the edge states guarantees that neither disorder nor roughness at the edge will backscatter edge electrons, as would happen in more standard one-dimensional systems. **Each filled LL, climbing above \( E_F \) at the edge, contribute one 1D IDEAL edge channel \( (T = 1) \).**

In the Hall bars, **the edge channels are equipotential** (dissipation-less edge channels play the role of “potential contacts”). For this reason, no need for precise positioning of voltage probes!
Visualization of Edge States

A scanning force microscope sensitive to electrostatics.

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the filling factor $i \equiv \frac{n_{2D}}{B/\Phi_0}$

– the number (not necessarily integer) of filled Landau levels.

Around integer values of $i$ the Hall voltage varies in the bulk of the 2DEG and the current is almost uniformly distributed over the whole sample width. At the quantum Hall plateaus, the Hall voltage variations, as well as the current, are confined near the sample edges.
Semi-Classical (oversimplified) Explanation

\[ V_{xy} \equiv V_H = V_{SD} = \frac{1}{e} (\mu_L - \mu_R) \]

\[ I_{xx} = I_{SD} = \frac{Q}{t} \quad \rho_{xy} \equiv \frac{V_{xy}}{I_{xx}} = \frac{V_{SD}}{I_{SD}} \]

How the current flows from S to D – unclear. But let’s ignore this.

There are \( b \) electrons in each edge channel. Because the wavefunctions shouldn’t overlap:

\[ b = \frac{L}{\lambda_F} = \frac{L}{\hbar} m v_F \]

The electrons move with the velocity \( v_F(\mu) \).

All \( b \) electrons will move across \( L \) in time

\[ t = \frac{L}{v_F} \]

The total SD current is due to the difference \( v_F(\mu_L) - v_F(\mu_R) \ll v_F \):

\[ I = \frac{e^2}{\hbar m} \frac{v_F(\mu_L)}{L} \left[ v_F(\mu_L) - v_F(\mu_R) \right] = \frac{e}{\hbar} m v_F \frac{v_F(\mu_L)}{L} \left[ v_F(\mu_L) - v_F(\mu_R) \right] \]

\[ \mu_L - \mu_R = \frac{1}{2} m \left[ v_F(\mu_L)^2 - v_F(\mu_R)^2 \right] \approx m v_F \left[ v_F(\mu_L) - v_F(\mu_R) \right] \]

\[ \rho_{xy} \equiv \frac{V_{xy}}{I_{xx}} = \frac{\hbar}{e^2} \]
Exact Quantization of $\rho_{xy}$

**Quantization of $\rho_{xy}$ is related to quantization of the electron charge and charge conservation!**

Consider the Corbino disk (there are contacts along all edges of the sample). In this geometry, if $B$ doesn’t depend on time, there is no Hall voltage ($E_{\phi} = 0$), the electric field is radial ($E_r$), the current circulates around the central disk-shaped contact ($I_{\phi}$).

Let’s assume that, in addition to a uniform field $B$, there is a *time-dependent* magnetic flux $\Phi(t)$ through the central disk. There will be non-zero $E_{\phi}$ such that

$$\int E_{\phi} dl = \left| \frac{d\Phi}{dt} \right|$$

Assume that the number of filled LLs is integer, $E_F$ is in the mobility gap (disorder!), and $\rho_{xx}$ ($\rho_{rr}$) and $\sigma_{xx}$ ($\sigma_{rr}$) vanish.

**B. Laughlin:** If $\Delta \Phi(t) = \Phi_0$, the system is the same as before (the Aharonov-Bohm phase is $2\pi$). The result of the flux change is that within each Landau level, each electron state exactly replaced its outer neighbor, and one state was added at the inside edge and disappeared at the outer edge. Thus, the net result is the transfer of the charge $\Delta Q = ne$ from one edge to the other when there are $n$ fully filled Landau levels.
The brilliance of Laughlin’s argument is that it still holds even if there are weak electron-electron interactions and (weak) disorder is present. At finite (but small) temperature the charge conservation is still an exact symmetry and therefore after a flux change by $\Phi_0$ only one electron per filled LL can travel from one edge to the other.

\[
\sigma_{xy} = \frac{I_r}{V_\phi} = \frac{ve}{\Phi_0} = \nu \frac{e^2}{h}
\]

\[
\rho_{xy} = \frac{h}{e^2} \times \left(\frac{1}{\nu}\right)
\]
Edge states in quantum Hall effect (by Bertrand Halperin)
https://www.youtube.com/watch?v=rQs12c-SieE&feature=youtu.be

Quantum Hall effect intro (by Ady Stern)
https://www.youtube.com/watch?v=QC3tQT7MD00

Quantum Hall effect summary (by Ady Stern)
https://www.youtube.com/watch?v=2u8_2isyi7o

Metrology

Quantum Metrology Triangle

Since the speed of light \( c \) is exactly defined quantity, the IQH effect provides one of the most accurate measurements of the fine structure constant \( \frac{e^2}{\hbar c} \) and gives a quantum standard for resistance.

The missing link, requires realization of Bloch oscillations in superconducting tunnel systems.

The current status (2012):

Summary

\[ \sigma_{xy} = \nu \frac{e^2}{h} \]
\[ \rho_{xy} = \frac{1}{\nu} \frac{h}{e^2} \]

\( \nu = \frac{n_{2D}}{B} 2\Phi_0 \) – the filling factor

Essential elements:

- Disorder allows for positioning \( E_F \) within the mobility gap – extended plateaus of \( \rho_{xy} \);
- Charge quantization and conservation results in \( \rho_{xy} = \frac{1}{\nu} \frac{h}{e^2} \).

“Grand unification”: disorder-induced localization, quantum channels, Landau levels, Aharonov-Bohm phase...