Lecture 11. Transport in 1D and 2D Structures

- 2D conductors
  - examples of 2D systems
  - 2D electron spectrum and DoS
  - why high $\mu$ is important

- 1D conductors
  - 1D quantum channels
  - two-terminal quantized conductance
  - four-terminal conductance
  - Landauer Formula
**Spectrum modification**: confinement of electrons in narrow potential wells (2D conductors, 1D quantum wires, or 0D quantum dots) results in large spacing between the energy levels corresponding to the motion in the confinement direction.

\[ \lambda_F = \frac{2\pi}{k_F} \]

**Novel transport effects**: when the structure dimensions become comparable to “classical” (the m.f.p. \( l \)) or quantum (\( L_\varphi, L_{1OC}, \lambda_F \)) length scales.

Quantum confinement \( W, d \sim \lambda_F \)

ballistic \( L < l, l_{in} \) diffusive \( L \gg l \)

coherent \( L_\varphi \gg l \), incoherent \( L_\varphi \sim l \)
Examples of 2D Structures: Si MOSFET

\[ n_{2D} = \frac{\varepsilon_{SiO_2} \varepsilon_0}{ed_{SiO_2}} (V_g - V_t) \]  
- 2D density of mobile electrons, \( V_t \) is the threshold voltage

\[ m^* \approx 0.2m_e \quad E_F \approx 7 \times \left( \frac{n_{2D}}{1 \times 10^{15} \text{ m}^{-2}} \right) \quad [K] \quad \lambda_F = \frac{h}{p} = \frac{h}{\sqrt{2E_F m^*}} \approx 10 \text{nm} \]

(two equivalent valleys, homework)

Mobility: \( \sim 3 \times 10^{-2} \text{m}^2/\text{Vs} \) at 300K, up to \( 3 \text{m}^2/\text{Vs} \) at low \( T \) (for metals \( \mu(300K) \sim 10^{-3} \text{m}^2/\text{Vs} \))

The m.f.p. \( l \): up to \( 0.1 - 0.3 \mu m \) at low \( T \)
Examples of 2D Structures: GaAs - AlGaAs Heterostructures

Charged donors are removed from the conducting channel – high mobility

Importance of weak disorder – slides 11,12

\[ m^* \approx 0.067m_e \]

Low-\( T \) mobility up to

\[ \mu \equiv \frac{e\tau}{m} \approx 3,000 \frac{m^2}{V} \cdot s \]

\[ \tau \approx 1ns \]

Low-\( T \) \( l \) – up to 100 \( \mu m \) (!)
Monolayers

- **hBN (insulator)**
- **MoS$_2$ (semiconductor)**
- **Black phosphorus (semiconductor)**
- **Graphene (semimetal)**

Energy (eV) diagrams for:
- **b** hBN: ~6 eV,
- **c** TMDC: ~1.0–2.5 eV,
- **d** BP: 0.3–2 eV,
- **e** Graphene: zero-gap
Simplest Model: Infinitely Deep Square Well

Electron motion is unrestricted in the $x, y$ plane of the well (the eigenstates are Bloch waves), and quantized in the transverse dimension ($z$).

Boundary conditions: $\psi(z = -a/2) = \psi(z = a/2) = 0$

$$
\psi_n(z) = \begin{cases} 
\sqrt{\frac{2}{a}} \cos \frac{n\pi z}{a}, & n \text{ even} \\
\sqrt{\frac{2}{a}} \sin \frac{n\pi z}{a}, & n \text{ odd}
\end{cases}
$$

Energy levels:

$$
\varepsilon_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 (n\pi/a)^2}{2m}
$$

$$
E_n(\vec{k}_\parallel) = \varepsilon_n + \frac{\hbar^2 k_\parallel^2}{2m}
$$

The first three energy levels and wave functions for a well in GaAs of width 10 nm.

Boundary conditions:

$$
\psi(z = -a/2) = \psi(z = a/2) = 0
$$

The first three energy levels and wave functions for a well in GaAs of width 10 nm.
2D Sub-Bands

\[ \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + V(z) \right] \Psi(x, y, z) = E\Psi(x, y, z) \]

\[ \left[ \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} - \frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] u(z) = Eu(z) \]

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] u(z) = \left[ E - \frac{\hbar^2 k_x^2}{2m} - \frac{\hbar^2 k_y^2}{2m} \right] u(z) \]

\[ \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] u_n(z) = \varepsilon_n u_n(z) \]

Free motion in the \( x, y \) plane
\[ \Psi(x, y, z) = e^{ik_x x} e^{ik_y y} u(z) \]

\[ \varepsilon = E - \frac{\hbar^2 k_x^2}{2m} - \frac{\hbar^2 k_y^2}{2m} \]

\[ \Psi_{\vec{k}_||,n}(x, y, z) = e^{i\vec{k}_|| \cdot \vec{r}} u_n(z) \]

\[ E_n(\vec{k}_||) = \varepsilon_n + \frac{\hbar^2 k_{\parallel}^2}{2m} \]

Electron energy in the \( n^{th} \) sub-band

Total energy in each sub-band includes the transverse energy \( \varepsilon_n \) and the kinetic energy of in-plane motion.
“Infinitely deep” well

Well of finite depth
The number of occupied sub-bands depends on the density of electrons and $T$:

$$n_{2D} = \sum_i n_i = \int_{-\infty}^{\infty} g_{2D}(\varepsilon)f(\varepsilon, E_F)d\varepsilon$$

$$T = 0 \quad n_i = g_{2D}(\varepsilon)(E_F - \varepsilon_i) = \frac{m}{\pi \hbar^2} (E_F - \varepsilon_i)$$

# of electrons in the $i^{th}$ sub-band

“bottom” of the $i^{th}$ sub-band

$$n_{2D} = \frac{m}{\pi \hbar^2} \sum_i (E_F - \varepsilon_i)$$ - total density of electrons in 2D structure

It is assumed above that $E_F > \varepsilon_i$, otherwise $n_i = 0$. 
What happens to $\mu$ if several sub-bands are filled

Low-mobility structures, donors are in the conduction channel
Si $\delta$-doped GaAs

$\mu_1 \approx 0.1 - 0.2 \frac{m^2}{V \cdot s}$

High-mobility structures, donors are removed from the conduction channel
GaAs/AlGaAs heterostructures

$\mu_2 \approx 0.7 \frac{m^2}{V \cdot s}$

decrease of $\mu$ due to inter-subband scattering


Strong Electron-Electron Interactions in 2D Structures

\[ E_F = \frac{(\hbar k_F)^2}{2m} = \frac{(\hbar)^2}{m} \pi n_{2D} \]

kinetic energy

\[ U = \frac{e^2}{4\pi \varepsilon \varepsilon_0 r} = \frac{e^2}{4\pi \varepsilon \varepsilon_0} \sqrt{\pi n_{2D}} \]

potential energy

\[ r_s \equiv \frac{U}{E_F} \propto \frac{1}{\sqrt{n_{2D}}} \]

- effective e-e interactions are stronger at lower electron densities

In 3D metals, \( r_s \sim 1 \). In 2D systems, interactions can change the ground state.
High $\mu$ : Path to New Physics

- **Integer Quantum Hall** (non-interacting electrons)
- **Fractional Quantum Hall**
- **Composite Fermions**
- **Physics of even-denominator fractions**
  - Non-abelian anyons?
  - Topological quantum computing?

**Mobility, $m^2/Vs$**

10 100 3000
Sub-Bands in 1D Conductors

The number of electron states that fit into a wire and can carry current:
Along $z$, the eigenstates are Bloch waves. Along $x, y$, the wave functions must be zero at the surface (the states with $k_x, k_y = 0$ are not allowed). Assume $L_x = L_y = W$. Only a single state survives if

$$\frac{\pi \sqrt{2}}{W} < k_F < \frac{\pi \sqrt{5}}{W} \quad \text{or} \quad \frac{\pi \sqrt{2}}{k_F} < W < \frac{\pi \sqrt{5}}{k_F}$$

Sub-bands correspond to each allowed pair $k_x, k_y$:

$$E_i = \varepsilon_i(k_x, k_y) + \frac{\hbar^2 k_z^2}{2m}$$
Analogy with the EM modes in a rectangular wave guide

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$k_z$
(a) The tip introduces a movable depletion region which scatters electron waves flowing from the quantum point contact (QPC). An image of electron flow is obtained by measuring the effect the tip has on the conductance as a function of tip position.

(b) Two-terminal conductance versus the QPC width controlled by the gate voltage. The insets below each step show simulations of the spatial pattern of electron flow for the transverse modes that contribute to the conductance.

(c-e) Experimental images of electron flow for the first three transverse modes of a QPC. The observed interference fringes spaced by half the Fermi wavelength demonstrate the coherence of electron flow.
What's wrong with these panels?
Crossover 1D – 2D
Observation of van Hove Singularities in the DoS of Carbon Nanotubes

\[ \frac{dj}{dV} \propto g_{\text{tip}} \times g_{\text{sample}} \]

**Figure 10** (a) The density of states for a semiconducting carbon nanotube as a function of energy. The Van Hove singularities are seen in the STM tunneling spectra of a nanotube shown in (b).
**Ballistic systems**: What happens when $L \ll \ell$? The formulae for conductivity that refer to the scattering time are clearly no longer valid since there is no scattering. Does this mean that the two-terminal resistance $= 0$?
Two-Terminal Ballistic Transport in 1D

Contacts are reservoirs of electrons at the chemical potentials shifted by $eV$ ($V$ is the applied voltage). Ballistic transport: all electrons that enter the wire from the left contact make it to the right contact and vice versa.

The dispersion relation for the electrons in the $i^{th}$ sub-band:

$$E_i(k_z) = \varepsilon_i + \frac{\hbar^2 k_z^2}{2m}$$

Current carried by one sub-band:

$$I = e\nu \frac{\delta n}{2} = e\frac{v}{2} \frac{\partial n}{\partial \varepsilon} \delta \varepsilon = \left\{ \begin{array}{c} g_{1D} = \frac{1}{\pi \hbar} \sqrt{\frac{2m}{\varepsilon}} \\ v = \sqrt{\frac{2\varepsilon}{m}} \end{array} \right\} = \frac{e}{\pi \hbar} (\mu_1 - \mu_2) = \frac{e^2}{\pi \hbar} V$$

1D ballistic transport - each sub-band carries the same current: $I_i = \frac{2e^2}{h} V$

Regardless of $L$, the conductance (not conductivity) is universal!
This result is **universal**, i.e. it applies to any form of the dispersion relation $\varepsilon(k)$, the only requirement is that $\mu_L - \mu_R \ll E_F$:

$$\mu_L - \mu_R = eV$$

Current carried by one sub-band:

$$I_i = \int_{\mu_R}^{\mu_L} ev_i(E) \left( \frac{1}{2} g_i^{1D}(\varepsilon) \right) d\varepsilon = \int_{k > 0} ev_i(\varepsilon) \left( \frac{2}{h\nu_i(\varepsilon)} \right) d\varepsilon = \frac{2e}{h} (\mu_L - \mu_R)$$

Remarkably, in 1D the product $\nu_i g_i^{1D}$ is universal *(regardless of the form of the dispersion relation $\varepsilon_i(k)$)*, and the current depends only on the voltage and fundamental constants.

$$I_i = \frac{2e^2}{h} V$$
Two-Terminal Conductance of 1D Systems

\[ I_i = \frac{2e^2}{h} V \]

The quantum of conductance:

\[ G_0 = \frac{2e^2}{h} \approx \frac{1}{12.9k\Omega} \]

(per two spin directions)

Note: conductance \( G \), not conductivity \( \sigma \).

Conductance is a **global** quantity.

If the spin degeneracy is lifted (e.g., quantum Hall effect) the conductance quantum is

\[ G_0 = \frac{e^2}{h} \approx \frac{1}{25.8k\Omega} \]

\[ \Rightarrow \textbf{von Klitzing constant} \quad \frac{h}{e^2} = 25,812,807,557 \ (18)\Omega \]

The two-terminal conductance of an **ideal** quantum wire is non-zero; it is proportional to the constriction width (i.e. the number of channels \( N = \frac{k_F W}{\pi} \)), but does not dependent on its length. The quantum of conductance is the smallest non-zero conductance of a ballistic 1D conductor. All ballistic conductors (multi-channel 1D, 2D, 3D) have conductance in multiples of the quantum conductance.
Two-Terminal Experiments with 1D wires


\[ \mu = 85 \frac{m^2}{V \cdot s} \]

\[ l > 4 \mu m \]

Quantization of the conductance in ballistic point contacts defined in the 2D electron gas of a GaAs-AlGaAs heterostructures by applying the voltage to a gate on top of the heterojunction.

**FIG. 2.** Quantized conductance of a quantum point contact at 0.6 K. The conductance was obtained from the measured resistance after subtraction of a constant series resistance of 400 Ω.

**FIG. 3.** Occupied electron states in the channel at two different gate voltages in the case of a current flow through the channel. In equilibrium the electron states are occupied up to the bulk Fermi energy \( E_F \). An applied voltage creates a difference \( eV = \mu_R - \mu_L \) between the electrochemical potentials of the reservoirs.
Fig. 7.21 A mechanically controlled break junction for observing conductance quantization in an Al QPC. The elastic substrate is bent by a pushing rod with a piezoelectric element. The thin Al bridge, fabricated by electron beam lithography, can be broken and reconnected for many cycles. Taken from [268].

Fig. 7.22 Top: Conductance as a function of the deformation time of a gold junction (an STM setup was used). The observed steps are usually not quantized in units of $2e^2/h$. Bottom: A histogram of many consecutive sweeps of the upper type, however, reveals that steps of the expected height dominate. Adapted from [62].
Four-Terminal Resistance of a Ballistic Quantum Wire

The electrons “injected” in the wire remain at $\mu_L$ and $\mu_R$ throughout the length of the wire (no inelastic scattering inside the wire). The steps in $\mu$ arise at the contacts with the reservoirs where electron scattering equilibrates left movers and right movers with the local electrochemical potential of the reservoir. The steps in $\mu_L$ and $\mu_R$ can be viewed as contact resistances to the wire, each having a value $R_{contact} = \frac{h}{2e^2}$. On the other hand, within the wire $\mu_L$ and $\mu_R$ are independent of position and the result of a four-terminal resistance measurement should be zero.

R. de Picciotto et al, 
Energy Dissipation in 1D wires

$P = I^2 R_Q$ implies that there should be power dissipation. Where is the energy dissipated? (not in the 1D ballistic conductor).

The energy is dissipated in the contacts: electrons are thermalized by loosing energy ($e - ph$ and $e - e$ collisions) in the contact. This is similar to tunneling.
The conductance $G$, being a global quantity, cannot be described in local terms (e.g., $\sigma = e^2 g(E_F) D$ is not applicable). In this case, the transport can be formulated as a quantum mechanical scattering problem.

$T_i$ – the transmission probability of the $i$-th channel (sub-band)

For $N$ sub-bands:

$$G(E_F) = \frac{2e^2}{h} \sum_i T_i(E_F) \approx \frac{2e^2}{h} N(E_F) \quad (T = 0)$$

$LF$ applies to “ballistic” wires, “diffusive” wires, and tunnel junctions:

$$\text{Resistance} = \frac{h}{2e^2 T_i} = \frac{h}{2e^2} \left(1 + \frac{1 - T_i}{T_i} \right) = \frac{h}{2e^2 T_i} + \frac{h}{2e^2} R_i$$

quantized contact resistance

scattering from barriers (zero for ballistic wires)
Non-zero Temperatures

Conductance at non-zero temperatures:

\[ f(E, E_F) = \frac{1}{\exp\left(\frac{E - E_F}{k_B T}\right) + 1} \]

\[ G(E_F, T) = \frac{2e^2}{h} \sum_i \int T_i(E) \left(-\frac{df}{dE}\right) dE \approx \frac{2e^2}{h} \sum_i f(E_i - E_F) \]

- thermal smearing of the conductance “staircase”

At even higher temperatures – incoherent transport, dephasing due to inelastic scattering, diffusion.

Figure: Conductance of a QPC with reflectionless contact at different temperatures, in which the transversal potential has been approximated with a harmonic potential (\( \hbar \omega_0 = 0.1 \ eV \)).
Summary

- Low dimensions affect the density of electronic states and mobility.
- Increase of $\mu$ paves the path for new physics of interacting electrons.
- Ballistic 1D conductors: universal quantized conductance.

Next time: Ibach and Luth, Panel XVI