Lecture 10. Coherent Transport in Disordered Electronic Systems

- Weak Localization Effects
- Strong (Anderson) Localization
- WL-SL crossover
Format: 5 problems, open lectures/books, can use calculators but no laptops.

Caution: books are less helpful than lectures (takes much more time to find relevant content).

Mostly conceptual problems. This means that you need to know the lectures and be able to quickly navigate them.

Make sure that you can quickly reproduce such results as $g(\varepsilon)$ and $E_F(n)$ in all dimensions.
Effective Dimensionality (with respect to \( \lambda = 2\pi / k \))

Dimensionality of electrons, phonons, etc. is the **effective** dimensionality of their \( k \)-space.

**Example**: phonons in a thin film

\[
\begin{align*}
g_{2D}(\varepsilon_{ph}) & \propto k \\
g_{3D}(\varepsilon_{ph}) & \propto k^2
\end{align*}
\]

Thus, the effective dimensionality of phonons can change with \( T \).
Consider a metal film with $c = 3 km/s$. How thin should be this film to be able to consider the phonons in the film as two-dimensional at $T = 1K$?

$$k_B T \sim \hbar c k \quad k \sim \frac{k_B T}{\hbar c}$$

Phonons can be considered as 2D if

$$k \lesssim \frac{\pi}{t} \quad \text{or} \quad t \lesssim \frac{\pi}{k}$$

$$t \lesssim \frac{\pi}{k} \sim \frac{\pi \hbar c}{k_B T} \approx 70nm$$
Disorder: Can It Transform Metal Into Insulator?

Ordered crystals:

Disordered materials:

Classification of disordered conductors is based on the resistivity at $T \to 0$.

Does the disorder-driven metal-insulator transition exist?
Interference Effects?

Electrons are waves. Why can we ignore the interference effects?

Consider diffusive motion between points A and B. The electron can move along different trajectories. The total probability to get from A to B:

\[ P_{AB} = \left| \sum_i A_i \right|^2 = \sum_i |A_i|^2 + \sum_{ij} |A_i A_j| \]

- different trajectories
- probabilities
- interference of amplitudes

The phase gain for each trajectory:

\[ \Delta \varphi = \frac{1}{\hbar} \int_A^B \vec{p}_F d\vec{l} \gg 1 \]

The interference term vanishes after averaging for most of the trajectories

\[ \left( \sum_{ij} |A_i A_j| \right) = 0 \]

As the result, the phase-related effects are washed away and the transport looks like a classical ("Sommerfeld-like") diffusive transport.

Can we observe the interference phenomena in transport?
Increase of Resistance at low $T$

The upturn of $R(T)$ is especially pronounced in low-dimensional disordered conductors.

Ultra-thin ($t = 4nm$) Ag film

$\Delta R(T)/(R(0) e^2/h)$

$A = 0.98$

$A = 0.12$

MG and V. Gubankov (1981)

Low-$\mu$ Si MOSFET

D. Bishop, D. Tsui, and R. Dynes (1980)

N. Giordano, PRB 22 (1980)

Ultra-thin ($t = 4nm$) Ag film

$e - ph$ scattering

AuPd nanowires
**Important exception**: the self-intersecting, loop-like trajectories (the time-reversed paths)

\[ \vec{p} \rightarrow -\vec{p} \quad d\vec{l} \rightarrow -d\vec{l} \quad \Delta \varphi = \frac{1}{\hbar} \int_{A}^{B} \vec{p}_F d\vec{l} \quad \text{the same for both trajectories} \]

\[ P_0 = |A_1|^2 + |A_2|^2 + 2ReA_1A_2 = 4|A_1|^2 \]

As a result, the probability to find an electron at (·)O **increases**, and the probability to get from A to B – **decreases. Coherent enhancement of back-scattering** results in the **weak localization** correction to the Drude conductivity.

Coherent enhancement of back-scattering is a general phenomenon, e.g. it was observed for photons in non-absorbing diffusive media.
**Diffusion and Diffusion Coefficient**

**Diffusion equation in 1D:**
\[ \frac{\partial \rho(x, t)}{\partial t} = D \frac{\partial^2 \rho(x, t)}{\partial t^2} \]

Assuming that \( N \) particles start from the origin at the initial time \( t = 0 \), the diffusion equation has the solution

\[ \rho(x, t) = \frac{N}{\sqrt{4\pi Dt}} e^{-x^2/(4Dt)} \]

\[ \langle x \rangle = \sqrt{\langle x^2 \rangle} = \sqrt{2Dt} \]

\[ D = \frac{l \cdot v_F}{d} \]

- the diffusion coefficient, 
\( d \) is the dimensionality of space

Units: \( m^2/s \). For a 3D metal with \( l = 10^{-8} m, v_F = 10^6 m/s \Rightarrow D = \frac{l \cdot v_F}{3} = 3 \times 10^{-3} m^2/s. \)

In 1s, an electron would move, on average, \(~ \sqrt{D}t ~ 0.05 m = 5 cm (~)\)

\[ \sigma = \frac{e^2 \tau n}{m} = \left( \tau = \frac{3D}{v_F^2} \right) = e^2 D \frac{3n}{2E_F} = e^2 D g_{3D}(E_F) \]

\[ \sigma = e^2 D g_{3D}(E_F) \]
Phase Coherence Length

Size of the relevant loops? The loops contribute to $\rho$ if the phase coherence is preserved while an electron travels along the loop.

Mechanisms of phase randomization: *inelastic* scattering (both $e - e$ and $e - ph$) and *spin* scattering.

![Graph showing $\tau_e$ vs. $T$ for thin films (l~d~10nm); labels for S - Hf, G - Ti.](image)

The coherence length:

$$L_\phi(T) = \sqrt{D\tau_\phi} \approx \sqrt{D\tau_\varepsilon}$$

$$L_\phi(T) = \sqrt{\frac{v_F^2}{2} \tau_\phi} \sim v_F \sqrt{\tau_\varepsilon}$$

$\tau$ is the momentum relaxation time (elastic scattering)

At low $T$ $\tau_\varepsilon$ can exceed $\tau$ by many orders of magnitude!

Estimate: $T = 1K$, $\tau_\varepsilon = 10^{-7}s$, $D = \frac{v_F^2}{2} = \frac{v_F d}{2} \approx 10^{-3}m^2/s$

$$L_\phi(T) \approx 10^{-5}m \gg l \approx 10^{-8}m$$

*elastic* scattering time

$$\tau \approx \frac{d}{v_F} \approx 10^{-14}s$$

$d$ – film thickness
Weak Localization Corrections to $\sigma$

The *dimensionality* of a conductor with respect to the WL effects:

$$L \leftrightarrow L_\varphi(T) = \sqrt{D\tau_\varphi}$$

$\Delta\sigma_{WL}$ is proportional to the *probability of self-intersecting* of an electron trajectory within the phase-coherent volume:

**3D:**

$$L_\varphi \ll \text{all dimensions}$$

$$\Delta\sigma_{WL}^{3D} \propto \int_\tau^{\tau_\varphi} \frac{\lambda_F^2 v_F}{(Dt)^{3/2}} dt \propto \frac{e^2}{\hbar} \left[ \text{const} + \frac{1}{L_\varphi(T)} \right]$$

**2D:**

$$d < L_\varphi$$

$$\Delta\sigma_{WL}^{2D} \propto \int_\tau^{\tau_\varphi} \frac{\lambda_F^2 v_F}{dDt} dt \propto -\frac{e^2}{\hbar} \ln \left[ \frac{L_\varphi(T)}{l} \right]$$

**Quasi-1D**

$$\lambda_F \ll d, W < L_\varphi$$

$$\Delta\sigma_{WL}^{1D} \propto \int_\tau^{\tau_\varphi} \frac{\lambda_F^2 v_F}{dW\sqrt{Dt}} dt \propto -\frac{e^2}{\hbar} L_\varphi(T)$$
No “metal” in 1D and 2D !!!

“Scaling Theory of Localization: Absence of Quantum Diffusion in Two Dimensions”
Over 6,000 citations, the famous “Gang of Four” paper

The probability of forming the loops depends on the disorder and dimensionality of the conductor – thus, the WL effects are stronger in low-dimensional strongly-disordered conductors.

The effective dimensionality with respect to the WL effects is determined by the coherence length \( L_\varphi = \sqrt{D \tau_\varphi} \) where \( D \) is the diffusion coefficient, \( \tau_\varphi \) is the coherence time. Since \( L_\varphi \) increases with approaching \( T = 0 \), even a “macroscopic” conductor may become low-dimensional with respect to WL at sufficiently low \( T \).

The WL effects are usually observed at low temperatures (typically, below \( \sim 20 – 30K \)). Indeed, the condition \( L_\varphi = \sqrt{D \tau_\varphi} \gg l \) should be met (otherwise, no “coherent” loops). With increasing \( T \), the rate of inelastic scattering increases (thus, \( L_\varphi \) decreases), and electrons loose their phase coherence at the scale comparable with \( l \).
Estimate for 2D Conductors

\[
\Delta \sigma_{WL}^{2D} = - \frac{e^2}{2\pi^2 \hbar} \ln \left[ \frac{L \varphi(T)}{l} \right]
\]

\[
\sigma = \sigma_{\text{Drude}} + \Delta \sigma_{WL} \quad \Delta \sigma_{WL} \ll \sigma_{\text{Drude}}
\]

Resistance “per square” \((L = W)\):
\[
R_{\square} = \frac{1}{\sigma_{\text{Drude}}} = \frac{\rho}{d} \quad [\Omega]
\]

\[
\frac{\Delta R_{WL}}{R_{\text{Drude}}} = - \frac{\Delta \sigma_{WL}}{\sigma_{\text{Drude}}}
\]

\[
\frac{\Delta R_{WL}}{R_{\text{Drude}}} = R_{\square} \frac{e^2}{2\pi^2 \hbar} \ln \left[ \frac{L \varphi(T)}{l} \right]
\]

\[
\frac{e^2}{2\pi^2 \hbar} \approx 1 \times 10^{-5} \Omega^{-1} \quad R_{\square} = 100 \Omega \quad L \varphi(T) \propto T^{-\alpha} \quad \alpha = 1.5 - 2
\]

\[
\Delta R_{WL}(1K) - \Delta R_{WL}(2K) = R_{\square} \frac{e^2}{2\pi^2 \hbar} \ln \left[ \frac{L \varphi(1K)}{L \varphi(2K)} \right] = R_{\square} \frac{\alpha e^2}{2\pi^2 \hbar} \ln 2 \approx 2 \cdot 10^{-3}
\]

Not much, but the WL effects are easily observable because \(\Delta R_{WL}\) depends \textbf{in a non-trivial way} on \(T\) and \(B\), in contrast to the «residual» resistivity. This provides a powerful method of measuring the inelastic scattering rate at low \(T\). Indeed, you can measure \(\tau_{\epsilon}\) even if this time is many orders of magnitude longer than \(\tau\)!
WL in external magnetic fields

Classically weak magnetic fields: no trajectory “bending”.

The Aharonov-Bohm phase acquired by electrons:

\[ \vec{p} \rightarrow \vec{p} - e\vec{A} \quad \Psi \rightarrow \Psi \exp \left( i \frac{e}{\hbar} \oint \vec{A} \cdot d\vec{l} \right) \]

\( \Delta \varphi \equiv 2 \frac{e}{\hbar} \oint \vec{A} \cdot d\vec{l} = \frac{2e}{\hbar} \int \vec{B} \cdot d\vec{a} = 2\pi \frac{\Phi}{\Phi_0} \) - the phase difference between CW and counter-CW trajectories

\( \Phi_0 = \frac{h}{2e} \approx 2 \times 10^{-15} T \cdot m^2 = 20 G \cdot \mu m^2 \)

- the magnetic flux quantum

\[ P_0 = |A_1|^2 + |A_2|^2 + 2|A_1||A_2| \cos(\Delta \varphi(B)) = 2A^2 \left[ 1 + \cos \left( 2\pi \frac{\Phi}{\Phi_0} \right) \right] \]

The characteristic length at which two interfering waves propagating along a loop in opposite directions acquire the phase difference \( \sim 2\pi \):

\[ L_B = \frac{\Phi_0}{\sqrt{2\pi B}} \]

- the magnetic length
Altshuler-Aronov-Spivak Effect in disordered conductors

\[ P_0 = 2A^2 \left[ 1 + \cos \left( \frac{2\pi \phi}{\Phi_0} \right) \right] \]

Period of oscillations: \( \Delta B = \frac{\Phi_0}{\pi r^2} \). Amplitude: \( \frac{\Delta R_{WL}}{R_{\text{Drude}}} = R \frac{e^2}{2\pi^2 \hbar} \).

Decay - because of a non-zero thickness of the film.
When the magnetic length $L_B$ becomes smaller than $L_\phi(T) \Rightarrow$ suppression of the WL contributions of the loops with $L_B < L < L_\phi$

**WL magnetoresistance:**
- negative (if spin-orbit scattering is weak)
- anisotropic in 1D and 2D (purely orbital)
- observed in classically weak $B$

WL MR measurements have been extensively used to study inelastic scattering.

$\Delta R/T = (R_\Box e^2/\pi h)$

$L_B = (dL_\phi)^{1/2}$

$B\perp, B\parallel$

Cu film
d=7 nm
$T=10$K

$\Delta R/R$


V. Pudalov, MG *et al.*, 2001

High-mobility Si MOSFET
AlGaAs/GaAs heterostructures, the electron density $n_{2D} = 2.8 \cdot 10^{15} m^{-2}$, mobility $\mu = 8.7 \, m^2/V \cdot s$, the electron diffusion constant $D = 0.085 \, \frac{m^2}{s}$.

The fundamental difference is the behavior at $T = 0$ (the “ground” state).

The definition of a “normal” metal: its conductivity must be finite at $T = 0$.

The quantum-coherent effects in the conductivity of disordered conductors: no matter how large $\sigma$ is at high $T$, with cooling a low-dimensional conductor should becomes an insulator.

This conclusion is for weakly interacting electrons. The situation may be more complicated in strongly correlated systems.

To better understand the crossover between weak and strong localization, let’s approach it from the insulating side.
Anderson Localization in 3D

Non-interacting electrons in a 3D ordered system \(\Rightarrow\) delocalized electron states

What happens if we crank up the disorder?

**P.W. Anderson** (’58): Quantum interference can completely suppress the diffusion of a particle in a random potential (disorder can localize a particle *despite tunneling*).

At \(T = 0\), there is a quantum phase transition in a 3D system at a critical strength of disorder, \(x_C\). At \(x > x_C\), the electron states at the Fermi level are localized and look like that:

One-electron wave functions in a disordered two-dimensional material. The left image depicts four different “multifractal” wave functions (red, blue, green and orange). An electron occupying one of these can conduct electricity. The right image depicts four Anderson localized wave functions. An electron occupying one of these is trapped close to the position of the peak and cannot conduct. (Yang-Zhi Chou and Matthew Foster/Rice U.)
Amorphous Solids: Mobility Edges

Electron states near the edges tail into what would have been the forbidden gap of the crystal; deep in the tail, the DoS falls off exponentially.

Localized–delocalized transition in 3D (Anderson transition) (L11) occurs near either band edge. The borderline between the extended and localized states is called a “mobility edge”.
3D Anderson Localization – Band Picture

Ordered crystals

- Metals, gapless
- Insulators, gapped

Disordered

- DoS “tails”

Ordered

- 1D, 2D

Disordered

- Any disorder
- 3D Anderson insulator

Strong disorder

1D, 2D

σ(T → 0) > 0

metal

σ(T) ∝ e \( \frac{E_C - E_F}{T} \)

insulator

σ = 0

“perfect” insulator
The localization length \( \xi_{\text{loc}} \) is given by \( \psi^2(r) \propto e^{-(r-r_0)/\xi_{\text{loc}}} \).

\( \xi \) depends on the \textit{dimensionality, disorder} and \textit{symmetry} of the system (e.g., \( \xi \) depends on the magnetic field).

### Quasi-1D conductors:

\( \lambda_F \ll d, W \ll \xi_{\text{loc}} \)

\[ \xi_{\text{1D}} \approx W k_F l \]

### 2D conductors:

\( t \ll \xi_{\text{loc}} \)

\[ \xi_{\text{2D}} = l \cdot \exp \left( \frac{\pi}{2} k_F l \right) \]

### 3D conductors:

\[ \xi_{\text{3D}} \propto (E_c - E)^{-\nu} \]
Finite Temperatures $\Rightarrow$ Dephasing Processes

All electronic states in 1D and 2D are localized. However, often these 1D and 2D systems are good conductors at high temperatures. Why?

If the disorder is not too strong and the temperature is not too low, the phase coherence will be lost before an electron has a chance to diffuse over $\xi$ - an electron will be frequently scattered between localized states, and it will diffuse ALMOST as if its wavefunction is not localized.

Length scale: the phase coherence length \[ L_\varphi = \sqrt{D\tau_\varphi} \]

Weak Localization: \[ \lambda_F \ll B < L_\varphi(T) \ll \xi \]

Strong Localization: \[ \lambda_F < B < \xi \ll \text{"} L_\varphi(T) \text{"} \]
**T-driven “WL ↔ SL” crossover in 1D**

**WL:** electrons lose phase coherence before they diffuse over $\xi_{loc}$ [$L_\phi (T) \ll \xi_{loc}$]. They don’t “feel” strong localization: their “localization envelope” each time is centered at a different location.

**SL:** electrons retain phase coherence at scales $\sim \xi_{loc}$

Crude estimate: the WL-SL crossover occurs when the Drude conductivity and the quantum correction to $\sigma$ become of the same order of magnitude.

2D:  
$$\sigma_{2D} = \frac{e^2}{\hbar} (k_F l) \quad \text{— “classical” value, L6}$$

$$\sigma_{2D} \sim \frac{e^2}{\hbar} \left( k_F l - \ln \frac{L_\phi (T)}{l} \right) \to 0 \quad \text{at} \quad L_\phi (T) \sim l \cdot e^{k_F l} \sim \xi_{2D}$$

quasi-1D:  
$$\sigma_{\text{quasi1D}} = \frac{e^2}{\hbar} (k_F l) W$$

$$\sigma_{\text{quasi1D}} \sim \frac{e^2}{\hbar} \left( k_F l W - L_\phi (T) \right) \to 0 \quad \text{at} \quad L_\phi (T) \sim k_F l W \sim \xi_{1D}$$
$T$-driven “WL $\Leftrightarrow$ SL” crossover in 1D (cont’d)

At the crossover ($T = T_\xi$):

$$L_\phi(T_\xi) \sim \xi$$

$$R_\xi \sim \frac{h}{e^2}$$

Summary

Weak Localization

$L_\varphi \ll \xi_{loc}$

WL-SL Crossover in 1D, 2D
Anderson MIT in 3D

Strong Localization

$L_\varphi \sim \xi_{loc}$

Disorder

quantum corrections
$
\Delta \sigma_{WL}(T, B)$, new length scales - $L_\varphi(T), L_B$

What is left untouched:
- universal conductance fluctuations in mesoscopic conductors;
- quantum corrections due to $e$-$e$ interaction effects – as important as single-electron WL corrections;
- many other spectacular quantum effects.

Exponentially strong
$
\sigma(T)$, hopping conductivity

What is left untouched:
Pretty much everything

Next time: Kittel Ch. 18