Conservation of charge, energy, and momentum

**Charge**

\[ Q = \text{charge} \]
\[ \rho = \text{charge density} \]
\[ J = \text{charge flux density = current density} \]

Differential conservation law:

\[ \nabla \cdot J = -\frac{\partial \rho}{\partial t} \]

Integral conservation law (volume \( V \) with surface \( S \)):

\[ \oint_S J \cdot da = -\frac{d}{dt} \int_V \rho \, d\tau = -\frac{d}{dt} Q^{(V)} \]

\[ X^{(V)} \text{ means } X_{\text{enc}}, \text{i.e., } X \text{ enclosed in volume } V. \]

**Energy**

\[ U_{\text{mech}} = \text{Particle mechanical energy (kinetic energy etc.)} \]
\[ u_{\text{mech}} = \text{Particle mechanical energy density} \]
\[ U_{\text{em}} = \text{Electromagnetic energy} = \int u_{\text{em}} \, d\tau \]

\[ u_{\text{em}} = \text{Electromagnetic energy density} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \]

This is called \( u \) in the book.

\[ S = \text{flux density of } U_{\text{em}} = \frac{1}{\mu_0} E \times B \]

Differential conservation law:

\[ \nabla \cdot S + E \cdot J = -\frac{\partial u_{\text{em}}}{\partial t} \]

Using \( E \cdot J = \frac{\partial}{\partial t} u_{\text{mech}} \) this can be rearranged:

or

\[ \frac{\partial}{\partial t} (u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot S \]

Integral conservation law (volume \( V \) with surface \( S \)):

\[ \oint_S S \cdot da + \frac{d}{dt} U_{\text{mech}}^{(V)} = -\frac{d}{dt} U_{\text{em}}^{(V)} \]

In the book \( \frac{d}{dt} U_{\text{mech}} \) is called \( W \), e.g., in Eq. (8.11)

or

\[ \frac{d}{dt} (U_{\text{mech}} + U_{\text{em}})^{(V)} = -\oint_S S \cdot da \]
**Momentum**

\[ \mathbf{p}_{\text{mech}} = \text{Particle momentum} \]

\[ \mathbf{P}_{\text{mech}} = \text{Particle momentum density} \]

\[ \mathbf{p}_{\text{em}} = \text{Electromagnetic momentum} \]

\[ \mathbf{P}_{\text{em}} = \text{Electromagnetic momentum density} = \mu_0 \varepsilon_0 \mathbf{S} = \varepsilon_0 \mathbf{E} \times \mathbf{B} \quad \text{Called } \mathbf{g} \text{ in the book.} \]

\[ -\mathbf{T} = \text{flux density of } \mathbf{p}_{\text{em}} \quad -\mathbf{T} = \mathbf{G} \text{ of my lecture. } \mathbf{T} \text{ is given by Eq. (8.17) in book.} \]

Differential conservation law:

\[ \frac{\partial}{\partial t} (\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{em}}) = -\nabla \cdot (-\mathbf{T}) = \nabla \cdot \mathbf{T} \]

Integral conservation law (volume \( V \) with surface \( S \)):

\[ \frac{d}{dt} (\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{em}}) (V) = -\int_S (-\mathbf{T}) \cdot da = \int_S \mathbf{T} \cdot da \quad \text{F in Eqs. (8.21-22) is } \frac{d}{dt} \mathbf{P}_{\text{mech}}. \]