Conservation of charge, energy, and momentum

Charge

\[ Q = \text{charge} \]
\[ \rho = \text{charge density} \]
\[ J = \text{charge flux density} = \text{current density} \]

Differential conservation law:
\[ \nabla \cdot J = -\frac{\partial \rho}{\partial t} \]

Integral conservation law (volume \( V \) with surface \( S \)):
\[ \oint_S J \cdot da = -\frac{d}{dt} \int_V \rho \, d\tau = -\frac{d}{dt} Q^{(V)} \]
\( X^{(V)} \) means \( X_{\text{enc}} \), i.e., \( X \) enclosed in volume \( V \).

Energy

\[ U_{\text{mech}} = \text{Particle mechanical energy (kinetic energy etc.)} \]
\[ u_{\text{mech}} = \text{Particle mechanical energy density} \]
\[ U_{\text{em}} = \text{Electromagnetic energy} = \int u_{\text{em}} \, d\tau \]
\[ u_{\text{em}} = \text{Electromagnetic energy density} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \]
This is called \( u \) in the book.

\[ S = \text{flux density of } U_{\text{em}} = \frac{1}{\mu_0} E \times B \]

Differential conservation law:
\[ \nabla \cdot S + E \cdot J = -\frac{\partial u_{\text{em}}}{\partial t} \]
Using \( E \cdot J = \frac{\partial}{\partial t} u_{\text{mech}} \) this can be rearranged:

\[ \frac{\partial}{\partial t} (u_{\text{mech}} + u_{\text{em}}) = -\nabla \cdot S \]

Integral conservation law (volume \( V \) with surface \( S \)):
\[ \oint_S S \cdot da + \frac{d}{dt} U_{\text{mech}}^{(V)} = -\frac{d}{dt} U_{\text{em}}^{(V)} \]
In the book \( \frac{d}{dt} U_{\text{mech}} \) is called \( W \), e.g., in Eq. (8.11)

or
\[ \frac{d}{dt} (U_{\text{mech}} + U_{\text{em}})^{(V)} = -\oint_S S \cdot da \]
Momentum

\( p_{\text{mech}} = \) Particle momentum

\( \mathcal{P}_{\text{mech}} = \) Particle momentum density

\( p_{\text{em}} = \) Electromagnetic momentum

\( \mathcal{P}_{\text{em}} = \) Electromagnetic momentum density = \( \mu_0 \epsilon_0 S = \epsilon_0 \mathbf{E} \times \mathbf{B} \) Called \( \mathbf{g} \) in the book.

\(-\mathbf{T} = \) flux density of \( p_{\text{em}} \)

\(-\mathbf{T} = \mathbf{G} \) of my lecture. \( \mathbf{T} \) is given by Eq. (8.17) in book.

Differential conservation law:

\[
\frac{\partial}{\partial t} (\mathcal{P}_{\text{mech}} + \mathcal{P}_{\text{em}}) = -\nabla \cdot (-\mathbf{T}) = \nabla \cdot \mathbf{T}
\]

Integral conservation law (volume \( V \) with surface \( S \)):

\[
\frac{d}{dt} (p_{\text{mech}} + p_{\text{em}}) (V) = -\int_S (-\mathbf{T}) \cdot d\mathbf{a} = \int_S \mathbf{T} \cdot d\mathbf{a}
\]

\( F \) in Eqs. (8.21-22) is \( \frac{d}{dt} p_{\text{mech}}(V) \).