Problem 1 (25 points)
An electromagnetic wave traveling in vacuum has electric field

\[ E(r, t) = E_0 (\hat{x} + \hat{y}) \sin(kz + \omega t). \]

(a) Is this wave linearly or circularly polarized? In what direction is it traveling?
(b) Write down the corresponding magnetic field.
(c) Find the electromagnetic energy density \( u_{em}(r, t) \) and Poynting vector \( S(r, t) \) associated with this wave (expressed in terms of \( E_0, \epsilon_0 \), and \( c \)).
Problem 2 (25 points)

Do all parts; later parts may not depend on earlier parts.

(a) A linearly polarized EM wave having intensity \( I = 48 \, \text{W/m}^2 \) is traveling in vacuum. What are amplitudes of the electric and magnetic fields, in V/m and T respectively?

(b) If this wave is normally incident on a dielectric medium with \( n = 7 \), what is the intensity of the wave that travels through this medium?

(c) Suppose the medium is dispersive with a dispersion relation \( k(\omega) = \gamma \omega^{3/2} \) for some constant \( \gamma \). If \( n = 7 \) at \( \omega = \omega_0 \), what is \( n \) at \( \omega = 2\omega_0 \)?

(d) Again assuming \( k(\omega) = \gamma \omega^{3/2} \), find the ratio \( v_g/v \) of the group velocity to the phase velocity of these waves.

(a) \( I = \frac{1}{2} \, c \, \varepsilon_0 \, E_0^2 \)

\[
E_0 = \left( 2I / (c\varepsilon_0) \right)^{1/2} = \left( 190 \, \text{V/m} \right)^{1/2}
\]

\[
B_0 = \frac{1}{c} \, E_0 = \left( 6.34 \times 10^{-7} \, \text{T} \right)
\]

(b) \( T = \frac{\varepsilon_0 n_2}{(n_1 + n_2)^2} = \frac{28}{64} = \frac{7}{16} \)

\[
I_T = \frac{7}{16} \times 48 \, \text{W/m}^2 = 21 \, \text{W/m}^2
\]

(c) \( n(\omega) = \frac{c \, k(\omega)}{\omega} = \frac{c \, \gamma \omega^{3/2}}{\omega} = c \, \sqrt{\omega}
\)

So \( n(2\omega_0) = \sqrt{2} \, n(\omega_0) = \sqrt{14} \approx 3.74 \)

(d) \( \frac{v_g}{v} = \frac{\omega}{k} \frac{k}{\omega} = \left( \frac{\partial k}{\partial \omega} \right)^{-1} \frac{k}{\omega} = \left( \frac{3}{2} \gamma \omega^{1/2} \right)^{-1} (\delta \omega^{1/2}) = \left( \frac{2}{3} \right) \)

(\*\*) You can avoid looking up the value of \( \varepsilon_0 \) with this trick:

\[
\left( \frac{\mu_0}{\varepsilon_0} \right)^{1/2} = 377 \, \Omega \quad \Rightarrow \quad \frac{1}{\mu_0} = \frac{1}{377 \, \varepsilon_0}
\]

\[
I = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \, E_0^2 = \frac{1}{2 \cdot 377 \, \varepsilon_0} \quad \Rightarrow \quad \varepsilon_0 = \frac{1}{2 \cdot 377 \, 48} \, \text{V/m}
\]

(\*\*) Some people tried to use \( I = \frac{1}{2} \, c \, \varepsilon_0 \, E_0^2 \) but this requires a knowledge of \( \varepsilon_0 \) inside the dielectric, which is not given.
**Problem 3** (25 points)

Suppose that a measurement of the conductivity of the water in Lake Michigan yields a value of \( \sigma = 2.4 \times 10^{-3} \text{ (Ω-m)}^{-1} \). Radio waves of frequency 120 MHz from a nearby microwave relay station are incident on the surface of the lake and penetrate into the lakewater. Assume \( \epsilon = \epsilon_0 \) and \( \mu = \mu_0 \) in the lakewater.

(a) Find the relaxation time \( \tau \) of the lakewater at this frequency.

(b) Find the approximate frequency, in Hz, at which the lakewater would cross over from being a “good conductor” (\( \omega < 1/\tau \)) to a “poor conductor” (\( \omega > 1/\tau \)), and show that we are in the latter regime here.

(c) The skin depth in a poor conductor is \( d = \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}} \). Calculate the skin depth for the conditions stated above, in meters. Also, explain what “skin depth” means physically.

\[\tau = \frac{\epsilon}{\sigma} = \frac{\epsilon_0}{\sigma} = \frac{8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2}{2.4 \times 10^{-3} \text{ (Ω-m)}^{-1}}\]

\[\tau = 3.69 \times 10^{-9} \text{ s}\]

(b) The crossover frequency is at \( \frac{\omega_c}{\Omega} = 1 \) so \( \omega_c = \frac{1}{\tau} = 2.71 \times 10^8 \text{ s}^{-1} \)

But \( \frac{\omega}{2\pi} = \frac{4.32 \times 10^7 \text{ Hz}}{4.3 \text{ MHz}} \)

We are at higher frequency (120 MHz) so it is a poor conductor.

\[d = \frac{2}{\sigma} \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{2}{2.4 \times 10^{-3} \text{ (Ω-m)}^{-1}} \times \frac{1}{377.52}\]

\[d = 2.21 \text{ m}\]
Problem 4 (25 points)

(a) Design a waveguide for which the cutoff frequency, below which no modes can propagate, is $\omega_{\text{cut}} = 4.8 \times 10^{10} \text{s}^{-1}$, and such that a second mode begins to propagate when $\omega$ exceeds $7.2 \times 10^{10} \text{s}^{-1}$. Tell me the shape and dimensions of the waveguide (in cm).

(b) At what frequency does the lowest TM mode begin to propagate in this waveguide?

(c) Find $v$ and $v_g$ (phase and group velocities) for waves that propagate at $\omega = \omega_{\text{cut}} \sqrt{2}$. (Reduce your answer to the form of a numerical multiplier times $c$.)

\[ a = \frac{c \pi \left(3 \times 10^8\right)}{\left(4.8 \times 10^{10} \text{s}^{-1}\right)} = 1.96 \times 10^{-2} \text{m} = 1.96 \text{cm} \]
\[ b = \frac{c \pi \left(3 \times 10^8\right)}{\left(7.2 \times 10^{10} \text{s}^{-1}\right)} = 1.51 \times 10^{-2} \text{m} = 1.31 \text{cm} \]

\[ \frac{1}{V} = \frac{1}{V_q} = \frac{1}{C} \frac{d}{d\omega} \left[ \frac{\omega}{\omega - \omega_m n^2} \right] = \frac{1}{c \left(1 - \left(\frac{\omega_m}{\omega}\right)^2\right)^{1/2}} = \frac{1}{c} \left(1 - \left(\frac{1}{2}\right)^2\right)^{-1/2} = \frac{3}{2} \]