Problem 1. (25 points)

Two wire loops lie next to each other as shown. The only things you know are that the resistance of the circular loop is $R_2$; the mutual inductance of the two loops is $M$; and the current in the square loop is some given $I_1(t)$.

(a) Suppose

\[ I_1(t) = I_0, \quad t < 0 \]
\[ I_1(t) = I_0 \left(1 - \frac{t^2}{\tau^2}\right), \quad 0 < t < \tau \]
\[ I_1(t) = 0, \quad t > \tau \]

(For example, perhaps the square loop had been connected to a battery until time $t = 0$, when some switch was opened over a time interval $\tau$.) Find the current $I_2(t)$ induced in the circular loop.

(b) Find the total charge $Q$ that flows in the circular loop during this time.

(c) Suppose all you know is that $I_1(t)$ falls monotonically from $I_0$ to 0 during some time. Is the answer to part (b) still the same? If so, show why.

\[ a) \quad I_2(t) = \frac{1}{R_2} \mathcal{E}_2(t) = -\frac{M}{R_2} \frac{dI_1}{dt} \]
\[ I_2(t) = 0 \text{ for } t < 0 \text{ or } t > \tau, \text{ but for } 0 < t < \tau: \]
\[ I_2(t) = -\frac{M}{R_2} \frac{d}{dt} \left[ I_0 \left(1 - \frac{t^2}{\tau^2}\right)\right] = \frac{2MI_0}{\tau^2 R_2} t \]

\[ b) \quad Q = \int_0^\tau I_2(t) \, dt = \frac{2MI_0}{\tau^2 R_2} \int_0^\tau t \, dt = \frac{MI_0}{R_2} \]

\[ c) \quad \text{What I wanted here was for you to recognize that an integral undoes a derivative:} \]
\[ Q = \int_0^\tau I_2(t) \, dt = -\frac{M}{R_2} \int_0^\tau \left( \frac{dI_1}{dt} \right) \, dt \]
\[ = -\frac{M}{R_2} \left( I_1(\tau) - I_1(0) \right) = \frac{M I_0}{R_2} \]

Some people got confused about "monotonically". The point is that if the flow is not monotonic, it could be possible to interpret the "charge flow" as $\int_0^\tau |I_2(t)| \, dt$, and I didn't want to treat this case. But I'm afraid I confused some of you unnecessarily.
**Problem 2.** (25 points)

Here is a useful contactless way of measuring the resistance of a coin. A solenoid (large cylinder with its axis along the $z$ axis) is driven with an oscillating current through the windings in such a way that the magnetic field inside is $\mathbf{B}(t) = B_0 \cos(\omega t) \hat{z}$.

(a) Find $\mathbf{E}(s)$ (magnitude and direction) inside the solenoid ($s$ is the distance from the axis).

(b) A coin is placed at the center of the solenoid as shown. We model it as a disk of radius $R$, thickness $d$, and conductivity $\sigma$. Find $\mathbf{J}(s)$ inside the disk.

(c) Find the total power $P(t)$ being dissipated to Joule heating inside the disk at instant $t$. (Hint: an integral is required.)

(d) Summarize by giving an expression for the average power dissipated, $\langle P \rangle$, in terms of $\sigma$, $B_0$, $\omega$, $R$, and $d$. (That is, $\langle P \rangle$ is the average of $P(t)$ over one cycle of the oscillation.)

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**Solution:**

(a) \[ E(s) = -\frac{\pi s^2}{2\pi} (-\omega) B_0 \sin(\omega t) \hat{z} \]

(b) \[ \mathbf{J} = \sigma \mathbf{E} = \frac{B_0 \omega s}{2} \sin(\omega t) \hat{z} \]

(c) The local power loss is \( \mathbf{v} \cdot \mathbf{E} \) (power per unit volume). The power is

\[
P(t) = \int V \mathbf{v} \cdot \mathbf{E} \, dV = \int_0^R \int_0^{2\pi} \int_0^d \mathbf{v} \cdot \mathbf{E} \, ds \, d\theta \, dz
\]

\[
= \pi B_0^2 \omega^2 \sigma \int_0^R \int_0^{2\pi} \int_0^d \mathbf{v} \cdot \mathbf{E} \, ds \, d\theta \, dz
\]

\[
= \pi B_0^2 \omega^2 \sigma \frac{R}{2} \frac{\sin^2 \omega t}{\sin^2 \omega t}
\]

(d) Average over one cycle $C_0 \Rightarrow$ factor of $\frac{1}{2}$

\[
[\frac{1}{2\pi} \int_0^{2\pi} \sin^2 \omega t \, dt = \frac{1}{2}] \Rightarrow \bar{P} = \frac{\pi B_0^2 \omega^2 \sigma R^2}{16}
\]
Problem 3. (25 points)

Suppose the electric field is known to be

\[ \mathbf{E}(r, t) = \frac{\alpha}{s} \sin(\omega t) \mathbf{\hat{z}} \]

independent of \( z \) and \( \phi \), where \( \alpha \) is a constant, and \( s \) is the radial coordinate of the \((s, \phi, z)\) cylindrical coordinate system. No actual currents flow; that is, \( \mathbf{J} = 0 \).

(a) Find the displacement current \( \mathbf{J}_d(r, t) \).

(b) Find the magnetic field induced by the displacement current at time \( t = 0 \).

**Hint**: Argue that \( \mathbf{B} = B(s) \mathbf{\hat{\phi}} \) (why?) and compute \( B(s) \).

\[ \mathbf{J}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \varepsilon_0 \frac{\alpha \omega}{s} \cos(\omega t) \mathbf{\hat{z}} \]

\[ \mathbf{J}_d \text{ acts as a source for } \mathbf{B} \text{ in the same way as a current along } \mathbf{\hat{z}}, \text{ so } \mathbf{B} = B(s) \mathbf{\hat{\phi}} \]

Set up Ampere’s loop:

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mathbf{J}_d \text{, and where at } t = 0 \text{, } \mathbf{J}_d = \frac{\varepsilon_0 \alpha \omega}{s} \mathbf{\hat{z}} \]

Many people wrote \( \mathbf{J}_d \text{, and } = \mathbf{J}_d(\pi s^2) \)

but this is wrong since \( \mathbf{J}_d = \mathbf{J}_d(s) \) depends on \( s \) ! An integral is needed:

\[ \mathbf{J}_d \text{, and } = \int_0^s \int_0^{2\pi} \mathbf{J}_d(s) \, ds \, d\phi \]

\[ = 2\pi \int_0^s \frac{\varepsilon_0 \alpha \omega}{s} \, ds = 2\pi \left( \varepsilon_0 \alpha \omega \right) s \]

\[ \text{LHS} = 2\pi s B(s) \]

\[ \Rightarrow \left[ \mathbf{B} = \mu_0 \varepsilon_0 \omega \mathbf{\phi} \right] \]
Problem 4. (25 points)

Consider two oppositely charged sheets, separated by a distance $d$ as shown; both sheets are moving to the right at the same velocity $v = v \hat{x}$.

(a) Find $E$ and $B$ between the plates.

(b) Find the electromagnetic energy density $u_{em}$ between the plates. Show that if $v \ll c$ then the magnetic contribution is negligible.

(c) Compute the Poynting vector $S$ between the plates.

(d) In the approximation $v \ll c$, compute the “apparent velocity” $v_{app} = S/u_{em}$ of the energy density between the plates.

(Hint: The answer to part (d) is a bit peculiar...)

\begin{align*}
\text{(Choose right-handed coordinates:)}
\end{align*}

\begin{align*}
\text{a) } E &= \frac{\sigma}{\varepsilon_0} \hat{x}, \\
\text{b) } B &= -\mu_0 \sigma v \hat{y},
\end{align*}

\begin{align*}
\text{Standard result for one sheet:} & \quad \begin{cases}
\uparrow E^\prime = \frac{\sigma}{\varepsilon_0}, \\
\downarrow E^\prime = -\frac{\sigma}{\varepsilon_0} \\
\uparrow B^\prime = -\frac{\mu_0 \varepsilon_0}{2} \\
\downarrow B^\prime = \frac{\mu_0 \varepsilon_0}{2}
\end{cases} \\
\text{Superposition for result between two sheets:} & \quad \begin{cases}
\uparrow E = \frac{\sigma}{\varepsilon_0}, \\
\downarrow E = -\frac{\sigma}{\varepsilon_0} \\
\uparrow B = -\frac{\mu_0 \varepsilon_0}{2} \\
\downarrow B = \frac{\mu_0 \varepsilon_0}{2}
\end{cases}
\end{align*}

\begin{align*}
\text{On use Gauss + Ampere. But OK to quote here as "standard results."}
\end{align*}

\begin{align*}
\text{b) } u_{em} &= \frac{\varepsilon_0}{2} E^2 + \frac{1}{\mu_0} B^2 \\
&= \frac{\sigma^2}{2 \varepsilon_0} \left( 1 + \varepsilon_0 \mu_0 v^2 \right) = \frac{\sigma^2}{2 \varepsilon_0} \left( 1 + \frac{v^2}{c^2} \right) = \frac{\sigma^2}{2 \varepsilon_0}
\end{align*}

\begin{align*}
\text{c) } \vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} \frac{\sigma}{\varepsilon_0} \mu_0 \sigma v (\hat{\varepsilon}_x \times \hat{y}) \\
&= \frac{\sigma^2 v}{\varepsilon_0} \hat{\varepsilon}_x
\end{align*}

\begin{align*}
\text{d) } v_{app} &= \frac{\vec{S}}{u_{em}} = \frac{\varepsilon_0}{\sigma^2} \frac{\sigma^2 v}{\varepsilon_0} = \sqrt{\varepsilon_0} \\
\text{(Strange factor of 2 is connected with mechanical tension in capacitor plates...)}
\end{align*}