Problem 1. (25 points)

A magnetic field of strength $B$ is pointing toward you out of the plane of the paper. The U-shaped metal rail is resistanceless, but a resistive metal bar (resistance $R$ between the two points of contact a distance $d$ apart) slides along the rails to the right at speed $v$.

(a) What current flows through the bar, and in what direction?

(b) What is the magnetic force on the bar, and in which direction?

(c) What is the power being dissipated to resistance in the bar? Does this match the mechanical power being applied to move the bar?

\[ \text{(a)} \quad E = -\frac{d\phi}{dt} = -B \frac{d}{dt} (\text{Area}) = -Bvd \]
\[ I = -\frac{Bvd}{R} \quad \text{(clockwise)} \]

\[ \text{(b)} \quad F = I \int d\vec{l} \times \vec{B} = I d B = -\frac{B^2 d^2 v}{R} \quad \text{(to left)} \]

\[ \text{(c)} \quad \text{Power} = I \cdot v = I \cdot E = \frac{B^2 v^2 d^2}{R} \]

Mechanical power applied to bar from outside:
\[ \vec{F} = \frac{\vec{B} \times \vec{v} d^2}{R} \]
\[ \text{Power} = \frac{dW}{dt} = \vec{F} \cdot \vec{v} = \frac{B^2 v^2 d^2}{R} \]
Problem 2. (25 points)

Loop 1 in the figure at right is made up of two long straight wires separated vertically by distance \( b \), and is so wide that the end pieces can be treated as being infinitely far away (as indicated by the dotted lines). Loop 2 is an \( a \times b \) rectangular loop spaced a distance \( b \) from Loop 1 as shown. Loop 2 has resistance \( R_2 \).

Note: The two parts of this question are somewhat independent. If you have trouble with Part (a), go on to Part (b), expressing your answer there in terms of a variable \( M \).

(a) Find the mutual inductance \( M \) between the two loops. (Hint: You should actually be able to make \( b \) fall out of your answer. For full credit, express your answer this way.)

(b) Suppose the current in the bottom loop is time-dependent with \( I_1 = I_1(t) = I_0 \exp(-\lambda t) \), where \( I_0 \) and \( \lambda \) are positive constants. Find the magnitude and direction of the current \( I_2 \) induced in Loop 2.
Problem 3. (25 points)

A parallel-plate capacitor has circular plates of radius $R$ a distance $d$ apart (neglect fringing fields), and is fed by wires carrying a current $I(t) = \gamma t^2$ where $\gamma$ is a constant. The charge on the plates is zero at $t = 0$.

(a) Find the charge $Q(t)$ on the bottom plate.
(b) Find the electric field $E(t)$ between the plates.
(c) Find the displacement current $J_d(t)$ between the plates.
(d) Find the induced magnetic field $B(s,t)$ (magnitude and direction) for $s < R$ between the plates, neglecting fringing-field effects ($s$ is the distance from the axis).

(a) $Q(t) = \int_0^t I(t')\,dt' = \int_0^t \gamma t'^2\,dt' = \frac{\gamma}{3} t^3$

(b) $\sigma(t) = \frac{Q(t)}{\pi R^2} = \frac{\gamma}{3\pi R^2} t^3$
$E(t) = \frac{\sigma(t)}{\epsilon_0} = \frac{\gamma}{3\pi \epsilon_0 R^2} t^3$

(c) $J_d = \epsilon_0 \frac{dE}{dt} = \frac{\gamma}{\pi R^2} t^2$

(d) Ampere Loop of radius $s$:

$\mathbf{B} = B \hat{\phi}$

$2\pi s B \phi = \mu_0 I_d \sin \phi$

$= \mu_0 \left(\pi s^2\right) J_d$

$B \phi = \frac{\mu_0 s}{2} J_d \Rightarrow \frac{B \phi}{\mu_0 s J_d} = \frac{1}{\frac{\mu_0 s^2}{2\pi R^2}} = \frac{1}{\left(\frac{\pi R^2}{s^2}\right)}$
Problem 4. (25 points)

An infinitely long cylinder of radius \( R \) is centered on the \( z \) axis and contains a magnetic field that grows with time according to

\[
B(s, t) = \begin{cases} 
  \alpha s^2 e^{t/\tau} \hat{z} & \text{for } s < R \\
  0 & \text{otherwise}
\end{cases}
\]

where \( \alpha \) and \( \tau \) are constants, \( s \) is distance from the axis, and \( t \) is time. No charges are present. As a function of \( t \):

(a) Find the resulting electric field (magnitude and direction) a distance \( s \) from the axis, both for \( s < R \) and \( s > R \).

(b) Find the Poynting vector \( \mathbf{S} \) a distance \( s \) from the axis, both for \( s < R \) and \( s > R \).

\[\mathbf{S} = \mathbf{E} \times \mathbf{B} \]

energy flows in (to build up field energy \( \frac{1}{2\mu_0} B^2 \)).