**Problem 1** (25 points)
A metal bar slides frictionlessly on two parallel conducting rails a distance \( l \) apart as shown, with a resistor \( R \) connecting the two rails. A constant and uniform magnetic field \( B \) points into the page everywhere.

(a) If the bar slides to the right at speed \( v \), what current flows through the resistor? In what direction?

(b) What is the magnetic force on the bar, and in what direction?

(c) What is the power being dissipated in the resistor?

(d) Are your results consistent with conservation of energy? Explain.

\[ a = \alpha \Rightarrow \frac{dv}{dt} = kv \]

\[ |E| = \frac{dv}{dt} = Blv \]

\[ I = \frac{E}{R} = \frac{Blv}{R} \]

**Answers:**
- **a)** 
  \[ I = \frac{E}{R} = \frac{Blv}{R} \]
  \[ \text{Signs: if } \text{v} \text{ is out of page, then } B = -B \hat{n}, \text{ and} \]
  \[ \text{positive } \varepsilon \text{ is counterclockwise.} \]
  \[ \varepsilon = -
  \]

- **b)** 
  \[ F = \vec{I} \times \vec{B} = -IldB \hat{x} = \left( \frac{Blv}{R} \right) lB = \frac{l^2 B^2 v}{R} \text{ to left} \]

- **c)** 
  \[ \text{Power} = \sum \varepsilon = \frac{l^2 B^2 v^2}{R} \]

- **d)** 
  \[ \text{Power (mechanical) applied to bar} = \vec{F} \cdot \vec{v} = \frac{l^2 B^2 v^2}{R} \text{ to left} \]
  
  So mechanical work is dissipated to heat in the electrical resistor.
Problem 2 (25 points)

A square loop of dimensions $d \times d$ is centered inside a rectangular loop of dimensions $3d \times l$ as shown. (The loops are coplanar.) In the limit that $l \to \infty$,

(a) Find the mutual inductance $M$ between the two loops.

(b) If the current in the square loop is $I = I_0 e^{-\gamma t}$, find the induced EMF in the rectangular loop as a function of time.

\[ a) \quad I_1 = \text{current in outer loop} \]
\[ I_2 = \text{current in inner loop} \]

\[ B(s) = \frac{\mu_0 I}{2\pi s} \hat{\Phi} \quad \text{for mag. field due to long straight wire.} \]

\[ \text{Flux from left wire} = \int_A B \cdot d\Phi = d \int_0^{2d} \frac{\mu_0 I_1}{2\pi s} ds \]
\[ = \frac{\mu_0 d I_1}{2\pi} \left[ \ln(2d) - \ln(d) \right] = \frac{\mu_0 d \ln(2)}{2\pi} I_1 \]

\[ \text{Flux from right wire the same} \]
\[ M = \frac{\text{Flux}}{I_1} = \frac{\mu_0 d \ln(2)}{2\pi} \]

\[ b) \quad i_2(t) = I_0 e^{-\gamma t} \]
\[ \xi_1(t) = -M \frac{di_2}{dt} \]
\[ = MI_0 \gamma e^{-\gamma t} \]
\[ \xi_1(t) = \frac{\mu_0 I_0 d \gamma \ln(2)}{2\pi} e^{-\gamma t} \]
Problem 3 (25 points)

A parallel-plate capacitor has circular plates of radius $R$, the separation between the plates is $d$, and it is fed by wires carrying a current $I(t) = I_0 \cos(\omega t)$ as shown. (The charge on the plates is zero at $t = 0$.)

(a) Find the charge $Q(t)$ on the bottom plate.
(b) Find the electric field $E(t)$ between the plates.
(c) Find the displacement current $J_d(t)$ between the plates.
(d) Find the induced magnetic field $B(s, t)$ (magnitude and direction) for $s < R$ between the plates, neglecting fringing-field effects. ($s$ is the distance from the axis.)

\begin{align*}
\text{(a)} \quad Q(t) &= \int_0^t I \, dt = \frac{I_0}{\omega} \sin \omega t \\
\text{(b)} \quad E(t) &= \frac{Q}{\varepsilon_0} = \frac{I_0}{\pi R^2 \varepsilon_0} \sin \omega t \\
\text{(c)} \quad J_d &= \varepsilon_0 \frac{dE}{dt} = \frac{I_0 \omega \cos \omega t}{\pi R^2} = \frac{I_0 \cos \omega t}{\pi R^2} \\
\text{(d)} \quad \oint B \cdot d\ell &= \mu_0 I_d, \quad \omega c \\
2\pi s \, B(s) &= \frac{\mu_0 I_0 \omega \cos \omega t}{\pi R^2} \frac{s^2}{c^2} = \frac{\mu_0 I_0 \omega}{2 \pi R^2} \cos \omega t \\
\vec{B}(s) &= \frac{\mu_0 I_0 s}{2 \pi R^2} \cos \omega t \hat{\phi}
\end{align*}
Problem 4 (25 points)

An ideal parallel-plate capacitor with plates in the $x$-$z$ plane, surface area $A$, and separation $d$ stores a charge $Q$. This capacitor is placed inside an ideal solenoid with its axis along $z$, having $N$ turns per unit length of wire carrying current $I$.

For consistency of answers, please set things up so that the $E$ and $B$ fields are in the $+\hat{x}$, $+\hat{y}$, or $+\hat{z}$ direction.

(a) Compute the Poynting vector between the plates of the capacitor. (Neglect fringing fields.)

(b) The capacitor is now discharged. Assuming it is free to move, find its final momentum (magnitude and direction).

\[
\begin{align*}
\vec{E} &= \frac{1}{\varepsilon_0} \frac{Q}{A} \hat{y} \\
\vec{B} &= \mu_0 NI \hat{z} \\
\vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\varepsilon_0} \frac{QNI}{A} \hat{x}
\end{align*}
\]

(b) Momentum in fields was

\[
\vec{p}_{em} = \int \vec{S}_{em} \, d\tau \text{ over vol.} = (\varepsilon_0 \mu_0 \vec{S}) (Ad) = \mu_0 QNI d \hat{z}
\]

This was imparted to mechanical momentum of capacitor.
Your exam will be similar in length and format to Problems (1-4) above. But to help you study, here are a few more questions you could think about how to answer.

Problem A
A square loop (side $a$, resistance $R$) lies in the $xy$ plane, and a uniform magnetic field $B$ along the $z$ axis passes through the loop. Find the total charge $Q$ that flows if the loop is flipped $180^\circ$ about the $x$ axis.

Problem B
An aluminum disk of radius $R$ with its axis along $\hat{z}$ is rotating around this axis at angular velocity $\omega$, in the presence of a uniform magnetic field $\mathbf{B} = B\hat{z}$. Find the induced EMF between the center and edge of the disk.

Problem C
An ideal parallel-plate capacitor with plate area $A$ is being charged up by a current $I(t) = t^2$, starting with $Q=0$ on the plates at time $t=0$. Compute the displacement current $J_d$ between the capacitor plates as a function of time.

Problem D
Consider $\mathbf{E} = \cos(kx-\omega t)\hat{y}$ and $\mathbf{B} = \cos(kx-\omega t)\hat{z}$. Compute the electromagnetic energy density and the Poynting vector as a function of $x$ and $t$.

Problem E
“Maxwell’s equations have built-in charge conservation.” True or false? Explain.