Physics 386 Spring 2006 Exam 2 Review

9.2: Electromagnetic Waves in vacuum: Maxwell's equations with no sources:

$$\vec{\nabla} \bullet \vec{E} = 0 \quad \text{Gauss's Law} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{Gauss's Law for Mag.} \qquad \vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \quad \text{Ampere's Law}$$

Take $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t}$ and substitute Ampere's law to get: $\vec{\nabla}^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$. Similarly, $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \mu_o \varepsilon_o \frac{\partial(\vec{\nabla} \times \vec{E})}{\partial t}$ leads to $\vec{\nabla}^2 \vec{B} = \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2}$. This means that electromagnetic waves move at a speed $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$ which, of course, is the speed of light $c = 3 \times 10^8$ m/s. Monochromatic plane EM waves: $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(kz-\omega t)}$; $\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}$. These waves are transverse in the sense that **E** and **B** are perpendicular to the direction of motion. They are plane polarized in the sense that the surfaces of constant phase are planes. Faraday's law gives: $\tilde{\vec{B}} = \frac{1}{c}\left(\hat{k} \times \tilde{\vec{E}}_0 e^{i(kz-\omega t)}\right)$. For a plane wave that propagates in the direction defined by the k-vector, **k**, we have $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k} \bullet \vec{r} - \omega t)}$; $\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(\vec{k} \bullet \vec{r} - \omega t)}$; $\tilde{\vec{B}} = \frac{1}{c}\left(\hat{k} \times \hat{n}\right) \tilde{E}_0 e^{i(\vec{k} \bullet \vec{r} - \omega t)}$ where $\hat{n}$ is the polarization vector of the wave. **E** ⊥ **B**. Clearly $\hat{n} \bullet \hat{k} = 0$ For a linearly polarized wave, $\hat{n}$ is a constant vector.

With $u = \frac{1}{2}\left(\varepsilon_o E^2 + \frac{1}{\mu_o} B^2\right)$ and $B^2 = \frac{1}{c^2} E^2$ for EM waves, $u = \varepsilon_o E^2 = \varepsilon_o E_o^2 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$. The Poynting vector becomes $\vec{S} = c\varepsilon_o E_o^2 \cos^2(\vec{k} \bullet \vec{r} - \omega t)\hat{k} = cu\hat{k}$. The momentum density of the em wave is $\mathbf{p}_{em} = \varepsilon_o \mu_o \vec{S} = \frac{1}{c^2}\vec{S} = \frac{1}{c}\varepsilon_o E_o^2 \cos^2(\vec{k} \bullet \vec{r} - \omega t)\hat{k} = \frac{1}{c} u\hat{k}$.

Taking the time average of these quantities we get a factor of ½ from the $\cos^2$ term which yields $\langle u \rangle = \frac{1}{2}\varepsilon_o E^2$; $\left\langle \vec{S} \right\rangle = \frac{1}{2} c\varepsilon_o E_o^2 \hat{k}$ and $\langle \mathbf{p}_{em} \rangle = \frac{1}{2c}\varepsilon_o E_o^2 \hat{k}$. The intensity is the magnitude of the time averaged poynting vector $I = \frac{1}{2} c\varepsilon_o E_o^2$. Finally, the radiation pressure is related to the intensity by $P = I/c$.

9.3: Maxwell's equations in media: With no free charges or free currents:

$$\vec{\nabla} \bullet \vec{D} = 0 \quad \text{Gauss's Law} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{Gauss's Law for Mag.} \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \text{Ampere's Law}$$

For linear medium, $\vec{D} = \varepsilon\vec{E}$, $\vec{H} = \frac{1}{\mu}\vec{B}$. thus:

$$\vec{\nabla} \bullet \vec{E} = 0 \quad \text{Gauss's Law} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{Gauss's Law for Mag.} \qquad \vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \quad \text{Ampere's Law}$$

For EM waves in matter the velocity is $v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$ where $n = \sqrt{\frac{\mu\varepsilon}{\mu_o\varepsilon_o}} = \frac{c}{n} \cong \sqrt{\varepsilon_r}$ for $\mu \sim \mu_o$.
Therefore: $u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)$, $\vec{S} = \frac{1}{\mu}\left(\vec{E} \times \vec{B}\right)$, $I = \frac{1}{2}v\varepsilon E_o^2$

Reflection and Refraction: At the interface between two linear media, the boundary condition becomes:

$$\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp} \quad \text{(from Gauss's Law)} \qquad \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \quad \text{(from Faraday's Law)}$$

$$B_1^{\perp} = B_2^{\perp} \quad \text{(from Gauss's Law for Mag.)} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel} \quad \text{(from Ampere's Law)}$$

Reflection and transmission at normal incidence: Separate fields into incident, reflected, and transmitted:

$$\tilde{\vec{E}}_I = \tilde{E}_{0I} e^{i(k_1 z - \omega t)}\hat{\boldsymbol{x}}, \ \tilde{\vec{B}}_I = \frac{1}{v_1}\tilde{E}_{0I} e^{i(k_1 z - \omega t)}\hat{\boldsymbol{y}} \ ; \ \tilde{\vec{E}}_R = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)}\hat{\boldsymbol{x}}, \ \tilde{\vec{B}}_R = -\frac{1}{v_1}\tilde{E}_{0R} e^{i(-k_1 z - \omega t)}\hat{\boldsymbol{y}}$$

$\tilde{\vec{E}}_T = \tilde{E}_{0T} e^{i(k_2 z - \omega t)}\hat{\boldsymbol{x}}$, $\tilde{\vec{B}}_T = \frac{1}{v_2}\tilde{E}_{0T} e^{i(k_2 z - \omega t)}\hat{\boldsymbol{y}}$. Applying the boundary conditions at Z = 0:

$\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R} = \tilde{E}_{0T}$ and $\frac{1}{\mu_1}(\frac{1}{v_1}\tilde{\vec{E}}_{0I} - \frac{1}{v_1}\tilde{\vec{E}}_{0R}) = \frac{1}{\mu_2}(\frac{1}{v_2}\tilde{E}_{0T})$ which leads to the relationship between real amplitudes: $E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I}$ and $E_{0T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I}$. The intensities are related by:

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \text{ and } T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2} \text{ giving } R + T = 1$$

Reflection and transmission at oblique incidence:

$$\tilde{\vec{E}}_I = \tilde{\vec{E}}_{0I} e^{i(\vec{k}_I \bullet \vec{r} - \omega t)}, \ \tilde{\vec{B}}_I = \frac{1}{v_1}\left(\hat{k}_I \times \tilde{\vec{E}}_I\right) \ ; \ \tilde{\vec{E}}_R = \tilde{\vec{E}}_{0R} e^{i(\vec{k}_R \bullet \vec{r} - \omega t)}, \ \tilde{\vec{B}}_R = \frac{1}{v_1}\left(\hat{k}_R \times \tilde{\vec{E}}_R\right)$$

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T} e^{i(\vec{k}_T \bullet \vec{r} - \omega t)}, \ \tilde{\vec{B}}_T = \frac{1}{v_2}\left(\hat{k}_T \times \tilde{\vec{E}}_T\right)$$

Matching of the phases implies that: $k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$, which gives:
the Law of Reflection: $\theta_I = \theta_R$ and the **Law of Refraction** (Snell's Law): $n_1 \sin\theta_I = n_2 \sin\theta_T$

With z normal to the interface, which lies in the xy plane, the matching conditions at the boundary become:

$$\varepsilon_1(\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R})_z = \varepsilon_2(\tilde{\vec{E}}_{0T})_z \quad \text{(from Gauss's Law)}$$

$$(\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R})_{x,y} = (\tilde{\vec{E}}_{0T})_{x,y} \quad \text{(from Faraday's Law)}$$

$$(\tilde{\vec{B}}_{0I} + \tilde{\vec{B}}_{0R})_z = (\tilde{\vec{B}}_{0T})_z \quad \text{(from Gauss's Law for Mag.)}$$

$$\tfrac{1}{\mu_1}(\tilde{\vec{B}}_{0I} + \tilde{\vec{B}}_{0R})_z = \tfrac{1}{\mu_2}(\tilde{\vec{B}}_{0T})_z \quad \text{(from Ampere's Law)}$$

with $\alpha = \dfrac{\cos\theta_T}{\cos\theta_I}$ and $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$ we have: $\tilde{E}_{oR} = \left(\dfrac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{oI}$, $\tilde{E}_{oT} = \left(\dfrac{2}{\alpha + \beta}\right)\tilde{E}_{oI}$

These are the **Fresnel Equations**. Brewster's angle ($E_{oR} = 0$) when $\alpha = \beta$ or $\tan\theta_B \cong \dfrac{n_2}{n_1}$

$$T = \frac{I_T}{I_I} = \frac{\mu_1 v_1}{\mu_2 v_2}\left(\frac{E_{oT}}{E_{oI}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

9.4 Absorption and Dispersion: In a conductor we have $\boldsymbol{J} = \sigma\boldsymbol{E}$ which leads to Ampere's law written as: $\vec{\nabla}\times\vec{B} = \mu\varepsilon\dfrac{\partial\vec{E}}{\partial t} + \mu\sigma\vec{E}$ which leads to a wave equation of the form:

$\vec{\nabla}^2\vec{E} = \mu\varepsilon\dfrac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\dfrac{\partial\vec{E}}{\partial t}$ which have solutions of the form $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}$

where $\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$ → $\tilde{k} = k + i\kappa$ so we can write E as $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$

For a material (dielectric) whose electrons can be modeled as damped driven harmonic oscillators, we can solve for $\boldsymbol{x}(t)$ from $\boldsymbol{F} = m\boldsymbol{a}$, to get $\boldsymbol{p}(t) = q\boldsymbol{x}(t)$ and $\mathbf{P} = N\boldsymbol{p}(t)$ which in turn is proportional to $\boldsymbol{E}$, so we can write $\boldsymbol{P} = \varepsilon_o\chi\boldsymbol{E}$. Since again χ can be complex we have $k = k + i\kappa$ which again give $\boldsymbol{E}$ of the form: $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}$. The dielectric constant can be derived from the response of the electron being driven by an applied electric field. For the case of an electron modeled by a damped spring-mass system, this results in the complex polarization: $\tilde{P} = \dfrac{Nq^2}{m}\dfrac{\mu_1 v_1}{\mu_2 v_2}\left(\dfrac{1}{\omega_2^2 - \omega^2 - i\gamma\omega}\right)\tilde{E}$.

Potentials and Fields: With Maxwell's equations including sources (i.e. current and charge densities) we still have the same relationship between the vector potential and the magnetic field: $\vec{B} = \vec{\nabla} \times \vec{A}$ however the electric field is given by: $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$. These definitions allow for a gauge transformation: $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$; $V' = V - \frac{\partial \lambda}{\partial t}$, where λ is some scalar function of space and time: $\lambda = \lambda(\boldsymbol{r}, t)$.

Coulomb Gauge: Choose $\vec{\nabla} \bullet \vec{A} = 0$ and $\nabla^2 V = -\frac{\rho}{\varepsilon_o}$. This gives rise to the familiar form for V from electrostatics, but the equations for $\boldsymbol{A}$ are very complicated.

Lorentz Gauge: Choose $\vec{\nabla} \bullet \vec{A} = -\mu_o \varepsilon_o \frac{\partial V}{\partial t}$ which gives rise to the very symmetric set of equations: $\nabla^2 \vec{A} - \mu_o \varepsilon_o \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_o \vec{J}$; and $\nabla^2 V - \mu_o \varepsilon_o \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_o}$, which can be written using the d'Alembertian: $\Box^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$; $\Box^2 V = -\rho/\varepsilon_0$ where $\Box^2 = \nabla^2 - \mu_o \varepsilon_o \frac{\partial^2}{\partial t^2}$.

The solutions to these equations are: $V(r,t) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r}\,', t_r)}{\Re} d\tau'$ and

$A(\vec{r},t) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t_r)}{\Re} d\tau'$ where $t_r = t - \Re / c$ and $\Re = |\vec{r} - \vec{r}\,'|$. The fields associated with these potentials are the Jefimenko fileds:

$$\vec{E}(r,t) = \frac{1}{4\pi\varepsilon_o} \int \left[\frac{\rho(\vec{r}\,', t_r)}{\Re^2} \hat{\Re} + \frac{\dot{\rho}(\vec{r}\,', t_r)}{c\Re} \hat{\Re} + \frac{\dot{\vec{J}}(\vec{r}\,', t_r)}{\Re} \right] d\tau' \text{ and}$$

$$\vec{B}(r,t) = \frac{\mu_o}{4\pi} \int \left[\frac{\vec{J}(\vec{r}\,', t_r)}{\Re^2} + \frac{\dot{\vec{J}}(\vec{r}\,', t_r)}{c\Re} \right] \times \hat{\Re} d\tau'.$$

For **point charges** we have the Lienard-Wiechert potentials: $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_o} \frac{qc}{(\Re c - \vec{\Re} \bullet \vec{v})}$ and $\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi} \frac{qc\vec{v}}{(\Re c - \vec{\Re} \bullet \vec{v})}$ where **v** is the velocity. These equations give the fields from a moving charge: $\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o} \frac{\Re}{(\vec{\Re} \bullet \vec{u})^3} [(c^2 - v^2)\vec{u} + \vec{\Re} \times (\vec{u} \times \vec{a})]$ and $\vec{B}(\vec{r},t) = \frac{1}{c} \hat{\Re} \times \vec{E}(\vec{r},t)$ where $\vec{u} = c\hat{\Re} - \vec{v}$.

Radiation: electric dipole:

$V(r,\theta,t) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q_o\cos[\omega(t-\Re_+/c)]}{\Re_+} - \frac{q_o\cos[\omega(t-\Re_-/c)]}{\Re_-}\right]$ which, with the appropriate approximations gives $V(r,\theta,t) = -\frac{p_o\omega}{4\pi c\varepsilon_o}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t-r/c)]$ and

$\vec{A}(r,\theta,t) = -\frac{\mu_o p_o\omega}{4\pi r}\sin[\omega(t-r/c)]\hat{z}$ for a dipole of moment $\vec{p}_o = q_o d\hat{z}$.

These lead to fields $\vec{E} = -\frac{\mu_o p_o\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\theta}$ and

$\vec{B} = -\frac{\mu_o p_o\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\phi}$. This give a Poynting vector:

$\vec{S} = \frac{\mu_o}{c}\left\{\frac{p_o\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}\hat{r}$ and $\langle\vec{S}\rangle = \left(\frac{\mu_o p_o^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$ for the time averaged vector. The time averaged radiated power is $\langle P\rangle = \left(\frac{\mu_o p_o^2\omega^4}{12\pi c}\right)$.

Magnetic dipole: $\vec{A}(\vec{r},t) = \frac{\mu_o}{4\pi}\int\frac{Io\cos[\omega(t-\Re/c)]}{\Re}d\vec{l}$ finally giving

$\vec{A}(r,\theta,t) = -\frac{\mu_o m_o\omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t-r/c)]\hat{\phi}$; $\vec{E} = \frac{\mu_o m_o\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\phi}$;

$\vec{B} = -\frac{\mu_o m_o\omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\hat{\theta}$; $\langle\vec{S}\rangle = \left(\frac{\mu_o m_o^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$; $\langle P\rangle = \left(\frac{\mu_o m_o^2\omega^4}{12\pi c^3}\right)$

From arbitrary source: $\vec{E} = \frac{\mu_o}{4\pi r}[(\hat{r}\bullet\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}}] = \frac{\mu_o}{4\pi r}[(\hat{r}\times(\hat{r}-\ddot{\vec{p}})]$; $\vec{B} = -\frac{\mu_o}{4\pi rc}[\hat{r}\times\ddot{\vec{p}}]$

$\vec{E} \approx \frac{\mu_o\ddot{p}(t_o)}{4\pi}\left(\frac{\sin\theta}{r}\right)\hat{\theta}$; $\vec{B} \approx \frac{\mu_o\ddot{p}(t_o)}{4\pi c}\left(\frac{\sin\theta}{r}\right)\hat{\theta}$; $\vec{S} \approx \frac{\mu_o}{16\pi^2 c}[\ddot{p}(t_o)]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$;

$P \approx \frac{\mu_o}{6\pi c}\ddot{p}^2$.

Point Charge: $\vec{E} = \frac{\mu_o q}{4\pi\Re}[(\hat{\Re}\bullet\vec{a})\hat{\Re} - \vec{a}] = \frac{\mu_o}{4\pi\varepsilon_o c^2\Re}[(\hat{\Re}\times(\hat{\Re}-\vec{a})]$;

$\vec{S} \approx \frac{\mu_o q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\Re^2}\right)\hat{\Re}$; $P = \frac{\mu_o q^2 a^2}{6\pi c}$