Physics 386 Spring 2008   Exam 1: Friday Feb 29; In Class; Review

7.1 Electromotive Force: Ohm's Law: $\boldsymbol{J} = \sigma \boldsymbol{E}$; $V = IR$; Power: $P = VI = I^2R$; EMF: integral of force per unit charge around a loop. **Motional EMF**: $\mathcal{E} = \oint \vec{f}_{mag} \bullet d\vec{l}$; for closed loop, can express this in terms of flux penetrating loop: $\Phi = \int \vec{B} \bullet d\vec{a}$ over the open surface bound by the closed loop.   This leads to $\mathcal{E} = = -\frac{d\Phi}{dt}$.

7.2 Electromagnetic Induction: Faraday's Law: $\mathcal{E} = = -\frac{d\Phi}{dt}$ regardless of whether $\frac{d\Phi}{dt}$ is produced by motion of the loop through a nonuniform magnetic field, or by a time-dependent magnetic field passing through a fixed loop. This gives: $\oint_P \vec{E} \bullet d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a}$. Written in differential form we have: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. This latter equation is a statement linking the electric and magnetic fields locally, not just a statement about induced emf's. The sign in Faraday's law is a reflection of **Lenz's law**, paraphrased as: The emf induced by a change in flux is such that it would induce a current that creates a magnetic field that opposes that change in flux. Anaology with Ampere's Law: Ampere says $\vec{\nabla} \times B = \mu_o \vec{J}$ and the nature of the fields relies on $\vec{\nabla} \bullet \vec{B} = 0$. IF we have no charge density, then $\vec{\nabla} \bullet \vec{E} = 0$. In this case, for situations with sufficient symmertry, we can use Faraday's law to find the induced electric field just as we used Ampere's law to find the magnetic field from a current density. Here $-\frac{\partial \vec{B}}{\partial t}$ plays the role of $\mu_o \vec{J}$. So, when we used $\oint_P \vec{B} \bullet d\vec{l} = \mu_o I_{enc}$ for Ampere's law, we can apply a similar circuital law for the induced electric field: $\oint_P \vec{E} \bullet d\vec{l} = -\frac{d\Phi}{dt}$. REMEMBER, FOR ALL OF THESE RESULTS, WE ARE IN THE QUASISTATIC APPROXIMATION.

Inductance: Typically, the flux through a loop is due to a magnetic field that is generated by a current in a second loop. The geometric factor that links the flux in loop 2 to the current in loop 1 is the mutual inductance: $\Phi_2 = M_{21}I_1$. Similarly, a current in loop 2 will induce a flux through loop 1, i.e., $\Phi_1 = M_{12}I_2$. Moreover, the mutual inductance linking two loops is identical, that is: $M_{12} = M_{21}$. Since a time rate of change in Φ results in an induced emf in the loop, we have $\mathcal{E}_2 = -d\Phi_2/dt = - M_{21}dI_1/dt$. A loop in which the current changes also creates a changing magnetic flux through itself, therefore $\Phi = LI$ where L is the self-inductance of the loop.

The energy needed to establish a current I in a loop with self inductance L (and therefore the energy stored in that "energized" loop) is given by $W = (1/2)LI^2$. By manipulating relations, we can express this purely in a form that depends solely on the fields: $W = \frac{1}{2\mu_o}\left[\int B^2 d\tau\right] - \oint_S (\vec{A}\times\vec{B})\bullet d\vec{a}$. We associate the quantity $\frac{B^2}{2\mu_o}$ with the energy density of the magnetic field.

7.3 Maxwell's Equations; Putting it all together we have Maxwell's Equations in vacuum:

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_o} \text{ Gauss's Law} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ Faraday's Law}$$

$$\vec{\nabla}\bullet\vec{B} = 0 \text{ Gauss's Law for Mag.} \qquad \vec{\nabla}\times\vec{B} = \mu_o\vec{J} \text{ Ampere's Law}$$

Ampere's Law is of course incomplete as demonstrated by the famous example of $\oint_P \vec{B}\bullet d\vec{l} = \mu_o I_{enc}$ being inconsistent for a wire leading to a capacitor. We must amend Ampere's law with the displacement current $\vec{J}_d = \varepsilon_o\frac{\partial\vec{E}}{\partial t}$ with which we can re-write Ampere's Law as $\vec{\nabla}\times\vec{B} = \mu_o(\vec{J}+\vec{J}_d) = \mu_o\vec{J} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$. This new formulation means that a time varying electric field generates a magnetic field, analogous to Faraday's law which states that a time varying magnetic field creates an electric field. So, the full Maxwell's equations read:

$$\vec{\nabla}\bullet\vec{E} = \frac{\rho}{\varepsilon_o} \text{ Gauss's Law} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ Faraday's Law}$$

$$\vec{\nabla}\bullet\vec{B} = 0 \text{ Gauss's Law for Mag.} \qquad \vec{\nabla}\times\vec{B} = \mu_o\vec{J} + \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t} \text{ Ampere's Law}$$

Maxwell's Equations in Matter: In the above equations, ρ and $\boldsymbol{J}$ refer to the total charge density or current density, both bound an free as one may have in media. Recall that $\rho_b = -\vec{\nabla}\bullet\vec{P}$ and $\vec{J}_b = \vec{\nabla}\times\vec{M}$ define the bound charge density and current density. Furthermore, with $\boldsymbol{D} = \varepsilon_o\boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{B} = \mu_o\boldsymbol{H} + \boldsymbol{M}$ we can rewrite Maxwell's equations in terms of the <u>free</u> charge density and <u>free</u> current density:

$$\vec{\nabla}\bullet\vec{D} = \rho_f \text{ Gauss's Law} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \text{ Faraday's Law}$$

$$\vec{\nabla}\bullet\vec{B} = 0 \text{ Gauss's Law for Mag.} \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \text{ Ampere's Law}$$

The boundary conditions at the interface between two media can be summarized as:
$D_1^\perp - D_2^\perp = \sigma_f$; $E_1^\parallel - E_2^\parallel = 0$; $B_1^\perp - B_2^\perp = 0$; $H_1^\parallel - H_2^\parallel = \vec{K}_f\times\hat{n}$

8.1: Conservation of Charge and Energy: The continuity equation for charge is given by $\frac{\partial\rho}{\partial t} = -\nabla\bullet\vec{J}$. This can be interpreted as the rate of decrease in charge density in a volume is equal to the flux of current out of the volume. We can generate a similar equation for the change in energy density in a volume and relate it to the work done on charges in the volume: $\frac{dW}{dt} = -\frac{d}{dt}\int \frac{1}{2}\left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right)d\tau - \frac{1}{\mu_o}\oint_S \left(\vec{E}\times\vec{B}\right)\bullet d\vec{a}$. So, we associate W with the mechanical energy of the charges in the volume, the first term with the electromagnetic energy in the volume, and the final term as the flux of energy out of the volume. We define the Poynting vector: $\vec{S} = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)$ so we can rewrite the previous equation as $\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S}\bullet d\vec{a}$ or $\frac{d}{dt}\int_V \left(u_{mech} + u_{em}\right)d\tau = -\oint_S \vec{S}\bullet d\vec{a}$ which can be recast as a local equation: $\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) = -\nabla\bullet\vec{S}$ which is analogous to the continuity equation for charge stated above.

If we consider electromagnetic forces on charges, we can take the Lorentz force eqn: $\vec{F} = \int_V \rho\left(\vec{E} + \vec{v}\times\vec{B}\right)d\tau = \int_V \left(\rho\vec{E} + \vec{J}\times\vec{B}\right)d\tau$ and write a force/unit volume: $\vec{f} = \left(\rho\vec{E} + \vec{J}\times\vec{B}\right)$

We can manipulate this to show that

$$\vec{f} = \varepsilon_o\left[\left(\vec{\nabla}\bullet\vec{E}\right)\vec{E} + \left(\vec{E}\bullet\vec{\nabla}\right)\vec{E}\right] + \frac{1}{\mu_o}\left[\left(\vec{\nabla}\bullet\vec{B}\right)\vec{B} + \left(\vec{B}\bullet\vec{\nabla}\right)\vec{B}\right] - \frac{1}{2}\vec{\nabla}\left(\varepsilon_o E^2 + \frac{1}{\mu_o}B^2\right) - \varepsilon_o\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

which can be re-written as $\vec{f} = \vec{\nabla}\bullet\overset{\leftrightarrow}{T} - \varepsilon_o\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$ where we define the Maxwell Stress Tensor which has components: $T_{ij} = \varepsilon_o\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$. Recall the operational use of a tensor, in that the inner product of a tensor and a vector is a vector: $\left(\vec{a}\bullet\overset{\leftrightarrow}{T}\right)_j = \sum_{i=x,y,z} a_iT_{ij}$. Thus we can write $\vec{f} = \vec{\nabla}\bullet\overset{\leftrightarrow}{T} - \varepsilon_o\mu_o\frac{\partial\vec{S}}{\partial t}$ which gives the total force as $\vec{F} = \int_V \vec{f}d\tau = \oint\overset{\leftrightarrow}{T}\bullet d\vec{a} - \varepsilon_o\mu_o\frac{d}{dt}\int_V \vec{S}d\tau$. Recalling that we can interpret force as the time rate of change of momentum, we can write: $\frac{d\vec{p}_{mech}}{dt} = \oint\overset{\leftrightarrow}{T}\bullet d\vec{a} - \varepsilon_o\mu_o\frac{d}{dt}\int_V \vec{S}d\tau$ which leads to interpretation of the last term as the time rate of change of the momentum of the EM field $\vec{p}_{em} = \varepsilon_o\mu_o\int_V \vec{S}d\tau$. So we can define a momentum density in the EM field: $\mathbf{p}_{em} = \varepsilon_o\mu_o\vec{S}$ and write this in differential form: $\frac{\partial}{\partial t}(\mathbf{p}_{mech} + \mathbf{p}_{em}) = \vec{\nabla}\bullet\overset{\leftrightarrow}{T}$. So we identify $-\overset{\leftrightarrow}{T}$ as the momentum flux density. Finally, there is also an angular momentum of the em field: $\boldsymbol{l}_{em} = \boldsymbol{r}\times\mathbf{p}_{em}$.

9.1: Waves in one dimension: A one dimensional wave is a disturbance in a medium that propagates in time but does not change shape. If we consider functions $g(\zeta)$ and $h(\zeta)$ that are functions of one variable, then a general expression for a wave that propagates in the z-direction is: $f(z, t) = g(z - vt) + h(z + vt)$ where v is the velocity of propagation of the wave. Waves (or commonly called wave functions) are solutions of a wave equation. The wave equation relevant to EM waves is: $\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$, of which the functions *g* and *h* above are solutions.

The most important solutions to the wave equation are sinusoidal plane wave solutions: $f(z, t) = A\cos[k(z - vt) + \delta]$. $\lambda = 2\pi/k$; $T = 2\pi/kv$; frequency $f = 1/T = v/\lambda$; $\omega = 2\pi f = vk$ typically write wave as $f(z, t) = A\cos(kz - \omega t + \delta)$. Often use complex notation $\tilde{f} = \tilde{A}e^{i(kz-\omega t)}$ where the physical wave function *f* is: $f = \Re e\{\tilde{f}\} = \Re e\{\tilde{A}e^{i(kz-\omega t)}\}$.

9.2: Electromagnetic Waves in vacuum: Maxwell's equations with no sources:

$$\vec{\nabla} \bullet \vec{E} = 0 \quad \text{Gauss's Law} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \quad \text{Gauss's Law for Mag.} \qquad \vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} \quad \text{Ampere's Law}$$

Take $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{\partial(\vec{\nabla} \times \vec{B})}{\partial t}$ and substitute Ampere's law to get: $\vec{\nabla}^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$. Similarly, $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) = \mu_o \varepsilon_o \frac{\partial(\vec{\nabla} \times \vec{E})}{\partial t}$ leads to $\vec{\nabla}^2 \vec{B} = \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2}$. This means that electromagnetic waves move at a speed $c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$ which, of course, is the speed of light $c = 3 \times 10^8$ m/s. Monochromatic plane EM waves: $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(kz-\omega t)}$; $\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}$. These waves are transverse in the sense that **E** and **B** are perpendicular to the direction of motion. They are plane polarized in the sense that the surfaces of constant phase are planes. Faraday's law gives: $\tilde{\vec{B}} = \frac{1}{c}\left(\hat{k} \times \tilde{\vec{E}}_0 e^{i(kz-\omega t)}\right)$. For a plane wave that propagates in the direction defined by the k-vector, **k**, we have $\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k} \bullet \vec{r} - \omega t)}$; $\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(\vec{k} \bullet \vec{r} - \omega t)}$; $\tilde{\vec{B}} = \frac{1}{c}\left(\hat{k} \times \hat{n}\right)\tilde{E}_0 e^{i(\vec{k} \bullet \vec{r} - \omega t)}$ where $\hat{n}$ is the polarization vector of the wave. **E** ⊥ **B**. Clearly $\hat{n} \bullet \hat{k} = 0$ For a linearly polarized wave, $\hat{n}$ is a constant vector.

With $u = \frac{1}{2}\left(\varepsilon_o E^2 + \frac{1}{\mu_o} B^2\right)$ and $B^2 = \frac{1}{c^2} E^2$ for EM waves, $u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)$. The Poynting vector becomes $\vec{S} = c\varepsilon_o E_o^2 \cos^2(\vec{k} \bullet \vec{r} - \omega t)\hat{k} = cu\hat{k}$. The momentum density of the em wave is $\mathbf{p}_{\text{em}} = \varepsilon_o \mu_o \vec{S} = \frac{1}{c^2}\vec{S} = \frac{1}{c}\varepsilon_o E_o^2 \cos^2(\vec{k} \bullet \vec{r} - \omega t)\hat{k} = \frac{1}{c}u\hat{k}$.

Taking the time average of these quantities we get a factor of ½ from the $\cos^2$ term which yields $\langle u \rangle = \frac{1}{2}\varepsilon_0 E^2$; $\left\langle \vec{S} \right\rangle = \frac{1}{2}c\varepsilon_o E_o^2 \hat{k}$ and $\langle \mathbf{p}_{\text{em}} \rangle = \frac{1}{2c}\varepsilon_o E_o^2 \hat{k}$. The intensity is the magnitude of the time averaged poynting vector $I = \frac{1}{2}c\varepsilon_o E_o^2$. Finally, the radiation pressure is related to the intensity by $P = I/c$.