

Name: \_\_\_\_\_

Ground rules:

- Open book
- Open notes
- Calculators are allowed
- Write your answer directly on these sheets (continue onto back, if necessary)

**There are 8 questions; point assignments are given for each, for a total of 100 points. Do all problems. Pace yourself appropriately.**

Show your reasoning. A correct answer all by itself, without any indication of the reasoning leading to it, will not get full credit. However, if you may use the formulas for “standard cases” (for example, the magnetic field a certain distance from a line current) in your solution without deriving them.

Partial credit will be given. Do as many parts of a problem as possible. If you are stuck on the first part of a problem, you should still try to say something about the later parts if possible.

Be sure to check that you have done all parts of all questions.

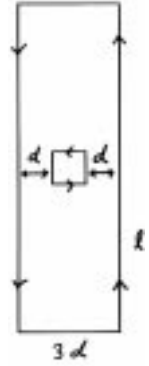
Feel free to raise your hand to ask a question.

Good luck, and have a good summer!

**Problem 1** (10 points)

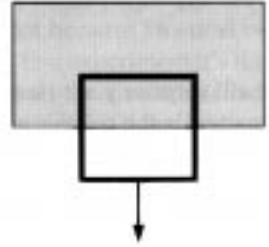
A square loop of dimensions  $d \times d$  is centered inside a rectangular loop of dimensions  $3d \times l$  as shown. (The loops are coplanar.) In the limit that  $l \rightarrow \infty$ ,

- Find the mutual inductance  $M$  between the two loops.
- If the current in the square loop is  $I = I_0 \sin(\omega t)$ , find the induced EMF in the rectangular loop as a function of time.



**Problem 2** (10 points)

A coil of wire contains  $N$  turns and is in the shape of a square of side  $b$  (so that the total length of wire is  $4Nb$ ). The resistance and mass of *one turn of wire* of length  $4b$  are  $R_0$  and  $m_0$ , respectively. The top of the coil is in a magnetic field of strength  $B$  pointing into the page (shaded region in figure), and is falling in a gravitational field  $g$ . Find the terminal velocity  $v$  of the coil (i.e., the velocity at which the electromagnetic and gravitational forces on the coil are in balance).



**Problem 3** (15 points)

A parallel-plate capacitor has circular plates of radius  $R$  and the separation between the plates is  $d$ . For  $t < 0$  the capacitor has remained uncharged, but at  $t = 0$  it is connected to a current source that delivers a current  $I(t) = \alpha t$  to the capacitor (where  $\alpha$  is some constant). Find, as a function of  $t$ :

- a) The charge on the capacitor.
- b) The  $\mathbf{E}$ -field between the plates.
- c) The displacement current  $\mathbf{J}_d$  between the plates.
- d) The induced  $\mathbf{B}$ -field between the plates at a distance  $s$  from the axis ( $s < R$  only).

You may neglect fringe effects, and work in the quasistatic approximation (that is, do not worry about a further  $E$  induced by  $dB/dt$ ). Be sure to specify the directions, as well as the magnitudes, of the vectors in your answers.

**Problem 4** (15 points)

Consider the potentials  $V(\mathbf{r}, t) = 0$  and  $\mathbf{A}(\mathbf{r}, t) = A_0 \hat{y} \sin(kx) \sin(\omega t)$  (representing so-called “standing electromagnetic waves”).

- a) Show that these potentials obey the Lorentz gauge condition.
- b) Show that these potentials satisfy the usual equations of motion for the potentials in vacuum (the “wave equation”) if  $k = \omega/c$ .
- c) Find  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{B}(\mathbf{r}, t)$ .
- d) Compute the Poynting vector  $\mathbf{S}(\mathbf{r}, t)$ , using the identity  $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$  to simplify your result. Show that the spatial average of  $\mathbf{S}$  is zero at all times, and that the time average of  $\mathbf{S}$  is zero at all places. (Thus, there is no net flux of energy;  $\mathbf{S}$  just shuttles energy around locally during the wave cycle.)

**Problem 5** (10 points)

A rectangular waveguide of dimensions  $6.0 \text{ cm} \times 2.4 \text{ cm}$  is excited at a frequency of  $\omega = 3.6 \times 10^{10} \text{ s}^{-1}$ .

- a) Show that there are just two TE modes that can propagate at this frequency, and identify them.
- b) Which mode can be used to communicate information more rapidly down the waveguide? Justify your answer.
- c) Find the wavevector  $k$  (in  $\text{cm}^{-1}$ ) and wavelength  $\lambda$  (in cm) of the (10) mode at the given frequency.

**Problem 6** (15 points) (probably the hardest question...)

A *neutral* current-carrying sheet lies in the  $x$ - $y$  plane and carries a uniform surface current density that was “turned on” at time  $t = 0$ :

$$\mathbf{K}(t) = \begin{cases} 0, & t < 0 \\ K_0 \hat{x}, & t > 0 \end{cases}$$

Important: If you have trouble with parts (a) and (b), *assume* the result quoted in part (b) and go on to parts (c) and (d).

a) For the point  $(0,0,z)$  a distance  $z$  above the plane, at time  $t > 0$ , show that  $t_r > 0$  at source points in the  $x$ - $y$  plane for which  $r < \sqrt{c^2 t^2 - z^2}$  (where  $r = \sqrt{x^2 + y^2}$ ).

b) Show that the retarded potential  $A_x(z, t) = \begin{cases} 0, & t < z/c \\ (\mu_0 K_0/2)(ct - z), & t > z/c \end{cases}$

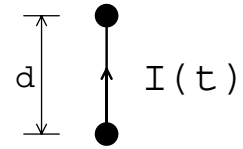
Hint: You should encounter an area integral of the form  $\int_0^{\sqrt{c^2 t^2 - z^2}} (\text{integrand}) 2\pi r dr$ , which can be evaluated via the substitution  $u = z^2 + r^2$ .

c) Also find  $A_y(\mathbf{r}, t)$ ,  $A_z(\mathbf{r}, t)$ , and  $V(\mathbf{r}, t)$ . (Remember, the sheet is neutral.)

d) Find  $\mathbf{E}(z, t)$  and  $\mathbf{B}(z, t)$ .

**Problem 7** (10 points)

The point of this problem is to get some feeling for whether the power loss to EM radiation is a significant factor for electronic devices, such as computer chips, that are not intended to radiate. For this purpose, consider again the model that we discussed for electric dipole radiation, in which a current  $I(t) = -q_0\omega \sin(\omega t)$  flows in a wire of length  $d$  connecting charges  $q_1(t) = q_0 \cos(\omega t)$  and  $q_2(t) = -q_0 \cos(\omega t)$ .



- Assuming that the connecting wire has resistance  $R$ , find the average power  $\langle P_{\text{res}} \rangle$  being dissipated to resistive heating.
- Compute the ratio  $\Gamma = \langle P_{\text{rad}} \rangle / \langle P_{\text{res}} \rangle$ , where the numerator is the average power lost to EM radiation. Express your answer in terms of  $d$  and  $\nu$ , where  $\omega = 2\pi\nu$ . (You may use a “standard result” for  $\langle P_{\text{rad}} \rangle$  without derivation.)
- Intel and AMD are just now coming out with 1 GHz CPU chips. Suppose, then, that an AC current flows at  $\nu = 1$  GHz through an interconnect (wire) of length 1 mm and resistance  $1 \text{ m}\Omega$ ; what is the ratio  $\Gamma$  for this case?

**Problem 8** (15 points)

In inertial frame  $S$ , one observes  $\mathbf{E} = E_0 \hat{y}$ ,  $\mathbf{B} = B_0 \hat{z}$ .

- a) Find explicitly the velocity  $\mathbf{v}$  (magnitude and direction) of a reference frame  $S'$  in which the electric field is zero.
- b) Find  $\mathbf{B}'$  in frame  $S'$ . (For consistency, please eliminate  $E_0$  and write your answer in terms of  $B_0$  and  $\gamma$ .)
- c) In a homework problem (Griffiths 12.46), you showed that  $I = E^2 - c^2 B^2$  is a relativistic invariant. Using your answers to parts (a) and (b), check explicitly that  $I$  is the same in the two frames.
- d) Evidently, if there really is a frame in which the electric field is zero, we must have  $I < 0$ . Suppose, on the contrary, that  $I > 0$ ; what was wrong with your answer to part (a) in this case?



