Physics 385          HW Solution #6          Fall 2018

1)  

a) Charges are given; pattern of V’s comes from symmetry.

b) When \( S_1 \) and \( S_3 \) are connected, a charge \( q/2 \) flows from \( S_1 \) to \( S_3 \), so that \( Q_1 = Q_3 = q/2 \), \( Q_2 = Q_4 = 0 \). (We also know \( V_1 = V_3 \) and \( V_2 = V_4 \).) Why? Because the problem clearly has a solution that is up-down symmetric. But by uniqueness, this must be the only solution.

c) Of course we still know the charges \( (Q_1 = Q_3 = q/2, Q_4 = q, Q_2=0) \) but there is no symmetry left and we can’t say much about the V’s.

d) There is a solution having complete symmetry \( (Q_1 = Q_2 = Q_3 = Q_4 = q/2 \text{ and } V_1 = V_2 = V_3 = V_4) \) and so this must be the solution.

2)  
I used a Fortran program:

```fortran
do i=0,16  
t=i*0.05  
going out along (100) cartesian direction  
rt=sqrt(t**2+1)  
v100=1/(1+t) + 1/(1-t) + 4/rt  
        0  0.000  6.0000  6.0000  
        1  0.050  6.0000  5.9999  
        2  0.100  6.0004  5.9979  
        3  0.150  6.0018  5.9988  
        4  0.200  6.0057  5.9689  
        5  0.250  6.0139  5.9275  
        6  0.300  6.0291  5.8569  
        7  0.350  6.0546  5.7583  

going out along (111) diagonal direction  
rtu=sqrt((t-1)**2+2*t**2)  
rtb=sqrt((t+1)**2+2*t**2)  
v111=3/rtu + 3/rtb  
        8  0.400  6.0949  5.6248  
        9  0.450  6.1555  5.4612  
       10  0.500  6.2444  5.2732  
       11  0.550  6.3722  5.0664  
       12  0.600  6.5550  4.8545  
       13  0.650  6.8170  4.6383  
       14  0.700  7.1985  4.4252  
       15  0.750  7.7714  4.2190  
       16  0.800  8.5790  4.0222
end do
end
```

Confined? Leaks out!

3)  

\( \rho \) \text{ SPHERICAL:}  
\[ \Box^2 V = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \frac{\partial V}{\partial \rho} \]  
\[ = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \left( \frac{A}{\rho^2} \right) \]  
\[ = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \left( \frac{B}{\rho^2} \right) \]  
\[ = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( -B \right) = 0 \]

\( \rho \) \text{ CYLINDRICAL:}  
\[ \Box^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial V}{\partial \rho} \]  
\[ = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \left( \frac{A}{\rho} \right) \]  
\[ = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \left( B \right) \]  
\[ = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( -B \right) = 0 \]
4) Place image charges $+2q$ at $z = -d$ and $-q$ at $z = -3d$. Total force on $+q$ is
\[
\mathbf{F} = \frac{q}{4\pi \varepsilon_0} \left[ \frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2} \right] \hat{z} = \frac{q^2}{4\pi \varepsilon_0 d^2} \left( -\frac{1}{2} + \frac{1}{8} - \frac{1}{36} \right) \hat{z} = \frac{1}{4\pi \varepsilon_0} \left( \frac{29q^2}{72d^2} \right) \hat{z}.
\]

5) The potential will be the same as for a cylinder filled with $+\rho$ above the plane and another filled with $-\rho$ below the plane. Recall that, by Gauss’s law, the fields (E or V) outside each cylinder are the same as those of a line charge of linear density $\lambda_{\text{eff}} = \pm (\pi a^2)\rho$. Thus, this problem is essentially the same as Griffiths 3.10, for which the solution is given below, except in your solution you should have substituted $\lambda$ by $\pi a^2 \rho$.

**Problem 3.10**

(a) Image problem: $\lambda$ above, $-\lambda$ below. Potential was found in Prob. 2.52:
\[
V(y, z) = \frac{2\lambda}{4\pi \varepsilon_0} \ln \left( \frac{s_-}{s_+} \right) = \frac{\lambda}{4\pi \varepsilon_0} \ln \left( \frac{s^2}{s_+^2} \right)
\]
\[
= \frac{\lambda}{4\pi \varepsilon_0} \ln \left( \frac{y^2 + (z + d)^2}{y^2 + (z - d)^2} \right).
\]

(b) $\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$. Here $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial z}$, evaluated at $z = 0$.
\[
\sigma(y) = -\varepsilon_0 \frac{\lambda}{4\pi \varepsilon_0} \left\{ \frac{1}{y^2 + (z + d)^2} 2(z + d) - \frac{1}{y^2 + (z - d)^2} 2(z - d) \right\} \bigg|_{z=0}
\]
\[
= \frac{2\lambda}{4\pi} \left\{ \frac{d}{y^2 + d^2} - \frac{-d}{y^2 + d^2} \right\} = \frac{-\lambda d}{\pi (y^2 + d^2)}.
\]

**Check:** Total charge induced on a strip of width $l$ parallel to the $y$ axis:
\[
q_{\text{ind}} = -\frac{l \lambda d}{\pi} \int_{-\infty}^{\infty} \frac{1}{y^2 + d^2} \, dy = -\frac{l \lambda d}{\pi} \left[ \frac{1}{d} \tan^{-1} \left( \frac{y}{d} \right) \right]_{-\infty}^{\infty} = \frac{l \lambda d}{\pi} \left[ \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right]
\]
\[
= -\lambda l. \quad \text{Therefore } \lambda_{\text{ind}} = -\lambda, \text{ as it should be.}
\]