Multipole expansion for the case of axial symmetry

This handout replaces Sec. 3.4.1 in the text, which your are not responsible for. However, you are responsible for what is here, including on homeworks, quizzes, and exams. You can bring this sheet to quizzes and exams.

Here we consider localized charge distributions \( \rho(r, \theta) \) with axial symmetry (i.e., independent of \( \phi \)), and try to find the leading behavior of the resulting \( V(r, \theta) \) far away.

Before getting started, let’s record the first few Legendre polynomials:

\[
\begin{align*}
P_0(\cos \theta) &= 1 \\
P_2(\cos \theta) &= \frac{3 \cos^2 \theta - 1}{2} \\
P_1(\cos \theta) &= \cos \theta \\
P_3(\cos \theta) &= \frac{5 \cos^3 \theta - 3 \cos \theta}{2}
\end{align*}
\]

Now, the general formula for the \( n \)’th multipole moment is

\[ I_n = \int r^n P_n(\cos \theta) \rho(r) \, d\tau \]

Let’s unpack this by writing the first three multipoles explicitly. Note that the “monopole” is nothing other than the net charge.

\[
\begin{align*}
n=0 & \quad \text{Monopole} & I_0 &= \int \rho(r) \, d\tau \\
n=1 & \quad \text{Dipole} & I_1 &= \int r \cos \theta \rho(r) \, d\tau = \int z \rho(r) \, d\tau \\
n=2 & \quad \text{Quadrupole} & I_2 &= \int r^2 P_2(\cos \theta) \rho(r) \, d\tau = \int \frac{3z^2 - r^2}{2} \rho(r) \, d\tau
\end{align*}
\]

The effect of the \( n \)’th multipole at large distances is

\[ V_n(r, \theta) = \frac{I_n}{4\pi \varepsilon_0} \frac{P_n(\cos \theta)}{r^{n+1}} \]

Explicitly:

\[
\begin{align*}
n=0 & \quad \text{Monopole} & V_0(r, \theta) &= \frac{I_0}{4\pi \varepsilon_0} \frac{1}{r} \\
n=1 & \quad \text{Dipole} & V_1(r, \theta) &= \frac{I_1}{4\pi \varepsilon_0} \frac{\cos \theta}{r^2} = \frac{I_1}{4\pi \varepsilon_0} \frac{z}{r^3} \\
n=2 & \quad \text{Quadrupole} & V_2(r, \theta) &= \frac{I_2}{4\pi \varepsilon_0} \frac{3 \cos^2 \theta - 1}{2r^3} = \frac{I_2}{4\pi \varepsilon_0} \frac{3z^2 - r^2}{2r^5}
\end{align*}
\]

Now, the procedure is like this:

- Find \( I_0 \) (calculate, or perhaps argue that it vanishes by symmetry). If \( I_0 \neq 0 \), answer is given by \( V_0(r, \theta) \). If \( I_0 = 0 \), go on to next step:
- Find \( I_1 \) (calculate, or perhaps argue that it vanishes by symmetry). If \( I_1 \neq 0 \), answer is given by \( V_1(r, \theta) \). If \( I_1 = 0 \), go on to next step:
- Find \( I_2 \) (calculate, or perhaps argue that it vanishes by symmetry). If \( I_2 \neq 0 \), answer is given by \( V_2(r, \theta) \). If \( I_2 = 0 \), go on to next step:
- Etc.