Problem 1. (25 points)

(a) The figure at right shows two concentric conducting spherical shells (hashed regions); the inner one extends from \( R_1 \) to \( R_2 \) and the outer one goes from \( R_3 \) to \( R_4 \). White spaces are vacuum. Find the capacitance between the two conductors. (Make sure your answer is written in terms of the correct radii.)

(b) Now interpreting the figure as showing two concentric conducting cylindrical shells in cross section, find the capacitance per unit length of this arrangement.

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a) To calculate \( C \), put charge \( Q \) on inner conductor and \(-Q\) on outer one. Then the charges look like

To the potential difference only depends on \( R_2 \) and \( R_3 \).

\[ E \text{ between } R_2 \text{ and } R_3 \text{ is } E_r = \frac{Q}{4\pi \epsilon_0} \frac{1}{r^2} \]

\[ V = \int_{R_2}^{R_3} E_r \, dr = \frac{Q}{4\pi \epsilon_0} \left(-\frac{1}{r}\right) \bigg|_{R_2}^{R_3} \]

\[ = \frac{Q}{4\pi \epsilon_0} \left( \frac{1}{R_2} - \frac{1}{R_3} \right) = \frac{Q}{4\pi \epsilon_0} \frac{R_3 - R_2}{R_2 R_3} \]

\[ C = \frac{Q}{V} \]

\[ C = 4\pi \epsilon_0 \frac{R_2 R_3}{R_3 - R_2} \]

b) \( E \) between \( R_2 \) and \( R_3 \) is \( E_5 = \frac{(Q/L)}{2\pi \epsilon_0} \frac{1}{5} \)

Putting charge \( Q \) on region of length \( L \)

\[ V = \frac{Q/L}{2\pi \epsilon_0} \int_{R_2}^{R_3} \frac{1}{5} \, ds = \frac{Q/L}{2\pi \epsilon_0} \ln \left( \frac{R_3}{R_2} \right) \]

\[ C = \frac{Q}{V} \]

\[ \frac{C}{L} = \frac{(Q/L)}{V} \]

\[ \frac{C}{L} = 2\pi \epsilon_0 \frac{1}{\ln \left( \frac{R_3}{R_2} \right)} \]
Problem 2. (25 points)

A positively charged styrofoam ball of radius $R$ carries a total charge $Q_0$ distributed uniformly through its interior (i.e., $Q_0 = \rho_0(4\pi R^3/3)$, but please express all answers in terms of $Q_0$, not $\rho_0$.) Its center is a distance $R + d$ below a conducting ceiling. (Note that the ball is not conducting.)

(a) Find the induced surface charge density $\sigma$ on the ceiling at point P just above the center of the ball.

(b) Find the electric field (magnitude and direction) at the center of the sphere.

(c) The weight of the ball is $mg$. How large must $Q_0$ be in order that the ball will stick to the ceiling (i.e., with $d = 0$) ?

\[ E_x(P) = \frac{Q_0}{4\pi\varepsilon_0 (R+d)^2} \]

\[ \sigma(P) = \varepsilon_0 E_x = -\varepsilon_0 E_x(P) = \frac{-Q_0}{2\pi (R+d)^2} \]

(b) \[ E_{\text{from self}} = 0 \]

\[ E_{\text{from } -Q_0} = \frac{Q_0}{4\pi\varepsilon_0 \frac{1}{2(R+d)^2} \hat{z}} \]

\[ E_z^2 = \frac{Q_0^2}{16\pi\varepsilon_0 \frac{1}{(R+d)^2}} > mg \text{ at } d = 0 : \]

\[ Q_0 > \frac{1}{16\pi\varepsilon_0 R^2 mg} \]

Note: uniformly charged sphere law

force law \[ F = \frac{Q_0^2}{4\pi\varepsilon_0 R^2} \]
Problem 3. (25 points)

You are given that the potential \( V(r, \theta) = V_0 \cos(\theta) \) at an inner radius \( r = a \), and it vanishes \( (V = 0) \) at an outer radius \( r = 2a \). Also, you know that there is no charge filling the shaded vacuum between these two radii. Find \( V(r, \theta) \) in this shaded region, i.e., for \( a < r < 2a \).

Boundary condition is \( V_0 \cos \theta = V_0 P_l(\cos \theta) \) with \( l = 1 \). so we try

\[
V(r, \theta) = (A r + \frac{B}{r^2}) \cos \theta
\]

Boundary condition at \( r = a \):

\[
V_0 = (Aa + \frac{B}{a^2}) \quad (1)
\]

Boundary condition at \( r = 2a \):

\[
0 = 2Aa + \frac{B}{4a^2} \quad (2)
\]

If you got this far, you get most credit. The rest is algebra:

\[
(2) \Rightarrow B = -8Aa^3
\]

\[
(1) \Rightarrow V_0 = Aa + (-8Aa^3)/a^2 = -8Aa \Rightarrow A = -V_0/7a
\]

\[
B = 8V_0a^2/7
\]

\[
V(r, \theta) = \frac{V_0}{7} (\frac{r}{a} + \frac{3a^2}{r^2}) \cos \theta
\]
Problem 4. (25 points)

Two rings of radius $a$ are centered around the $z$ axis with their centers at $z = b$ and $z = -b$.

(a) The top and bottom rings carry linear charge density $\lambda_0$ and $-\lambda_0$ respectively. What is the leading behavior of $V(r, \theta)$ far away?

(b) Now both rings carry the same linear charge density $\lambda_0$. Answer the same question again for this case.

(c) Suppose you are asked to modify the charge configuration of part (b) by adding one point charge $Q$ at a location of your choosing, in such a way that the leading behavior of $V$ far away is the quadrupolar one that decays as $1/r^3$. What would you choose for $Q$, and where would you place this charge? Explain.

(a) Charge neutral $\Rightarrow$ monopole $= 0$

Dipole:$ \mathbf{p}_z = \int \varphi z \, d\mathbf{S}$

$= b \left(2\pi a \lambda_0\right) + (-b) \left(2\pi a\right) \left(-\lambda_0\right)$

$= 4\pi ab \lambda_0$

$V(r, \theta) \overset{r \to \infty}{\approx} \frac{1}{4\pi \varepsilon_0} \frac{\mathbf{p}_z}{r^2} \cos \theta$

$= \frac{ab\lambda_0}{\varepsilon_0} \frac{1}{r^2} \cos \theta$

(b) Charge $= 2 \left(2\pi a \lambda_0\right) = 4\pi a \lambda_0 = 0$

$V(r, \theta) \overset{r \to \infty}{\approx} \frac{\mathbf{Q}}{4\pi \varepsilon_0} \frac{1}{r}$

$= \frac{\alpha \lambda_0}{\varepsilon_0} \frac{1}{r}$

(c) We can make the system neutral by adding $Q = -4\pi a \lambda_0$

To make the dipole vanish, place this at the origin $x = y = z = 0$. Then the symmetry is such that the dipole vanishes.