**Problem 1.** (25 points)

Conductor I is a sphere of radius $R_1$ with a cubical cavity at its center. It is centered inside Conductor II, which is a large cube with a spherical cavity of radius $R_2$. Starting from neutrality, a charge $+Q$ is added to Conductor I and a charge $-Q$ is added to Conductor II.

(a) Describe the resulting configuration of surface charge densities $\sigma$ on the four surfaces. **Hint:** Guess a simple solution in which the surface charges vanish on two of the four surfaces, and briefly justify it.

(b) Find the electric field in the vacuum region $R_1 < r < R_2$.

(c) Find the capacitance between Conductors I and II.

\[ a) \quad \sigma = 0 \text{ everywhere on each cubical surface}; \]
\[ \sigma = \frac{Q}{4\pi \varepsilon_0} \frac{r^2}{R_1^2} \text{ at } R_1; \quad \sigma = -\frac{Q}{4\pi \varepsilon_0} \frac{r^2}{R_2^2} \text{ at } R_2. \]

Then $E = 0$ in both regions I and II, and the total charges on I and II are correct, so by uniqueness this must be the solution.

\[ b) \quad E = \frac{Q}{4\pi \varepsilon_0} \frac{r}{r^2} \text{ by Gauss' Law}. \]

\[ c) \quad V_I - V_{II} = -\int_{R_1}^{R_2} E_r \, dr \]
\[ = \int_{R_1}^{R_2} \frac{Q}{4\pi \varepsilon_0} \frac{1}{r^2} \, dr \]
\[ = \frac{Q}{4\pi \varepsilon_0} \left[ \frac{1}{R_1} - \frac{1}{R_2} \right] = \frac{Q}{4\pi \varepsilon_0} \frac{R_2 - R_1}{R_1 R_2} \]

\[ C = \frac{Q}{V} = \frac{Q}{V_I - V_{II}} \Rightarrow \]
\[ C = \frac{4\pi \varepsilon_0}{R_2 - R_1} \frac{R_1 R_2}{R_2 - R_1} \]
Problem 2. (25 points)

A sphere of radius $R$ has potential $V(\theta) = V_0 (5 \cos^3 \theta - 3 \cos \theta)$ on its surface, and its interior is empty of all charge. Find $V(r, \theta)$ inside.

(Note: A list of Legendre polynomials is provided near the bottom of the exam cover page.)

\[ V(\theta) = 2V_0 P_3(\cos \theta) \quad \text{at} \quad r = R \]

Guess

\[ V(r, \theta) = \left[ A r^3 + \frac{B}{r^4} \right] P_3(\cos \theta) \]

Since want solution well-behaved at $r = 0$, set $B = 0$.

\[ V(r, \theta) = A r^3 P_3(\cos \theta) = 2V_0 P_3(\cos \theta) \]

\[ \Rightarrow A = \frac{2V_0}{R^3} \]

\[ V(r, \theta) = \frac{2V_0}{R^3} r^3 P_3(\cos \theta) \]

\[ V(r, \theta) = V_0 \frac{r^3}{R^3} (5 \cos^3 \theta - 3 \cos \theta) \]
Problem 3. (25 points)

A disk of radius $a$ is centered on the origin and lies in the $x$-$y$ plane, and it carries a variable surface charge density $\sigma(r) = \sigma_0 \left(1 - \frac{3r}{2a}\right)$. What is the leading behavior of $V(r, \theta)$ at large $r$?

\[
I_0 = \frac{\pi \sigma_0}{2} \quad V(r, \theta) = \frac{\pi \sigma_0 a^4}{2} \left(\frac{1}{4\pi \varepsilon_0} \frac{1}{r^2} \frac{1}{r^2} P_2(\cos \theta)\right) = \frac{\sigma_0 a^4 P_2(\cos \theta)}{80 \varepsilon_0 r^3}
\]
Problem 4. (25 points)

A circular wire ring of radius $R$ carries a uniform line charge $\lambda$, and lies in a plane that is parallel to the surface of an (infinite) grounded conducting plate. The height of the ring above the surface of the plate is $h$.

(a) Find the potential $V$ at the center of the wire ring.

(b) Find the induced surface charge $\sigma$ at the point on the metal surface directly under the center of the ring.

Hint: Do the two parts of this problem independently; you don’t need the answer to (a) to get (b), or vice versa.

\[ V(\text{center of ring}) = V_{\text{due to } \lambda} + V_{\text{due to } -\lambda} \]

\[ V_{\text{due to } \lambda} = \frac{2\pi R \lambda}{4\pi\varepsilon_0 R} = \frac{\lambda}{2\varepsilon_0} \]

\[ V_{\text{due to } -\lambda} = \frac{-2\pi R \lambda}{2\pi\varepsilon_0 R^2} \frac{1}{\sqrt{h^2 + R^2}} \]

\[ V(\text{center of ring}) = \frac{\lambda}{2\varepsilon_0} \left[ \frac{1}{\sqrt{h^2 + R^2}} \right] \]

\[ E_2 \text{ (at origin) } = \frac{1}{4\pi\varepsilon_0} \left[ \int_{\text{ring}} \frac{\lambda\hat{\mathbf{r}}}{r^2} \, dl + \int_{\text{bottom ring}} \frac{(-\lambda)\hat{\mathbf{r}}}{r^2} \, dl \right] \cdot \hat{\mathbf{z}} \]

\[ = \frac{2\lambda}{4\pi\varepsilon_0} \int_{\text{ring}} \frac{\hat{\mathbf{z}}}{r^2} \, dl = \frac{2\lambda (2\pi R)}{4\pi\varepsilon_0} \frac{R^2}{R^2 + h^2} \]

\[ \sigma = \varepsilon_0 E_{\perp} = \varepsilon_0 E_z \]

\[ \sigma = \frac{-R^2 \lambda}{(R^2 + h^2)^{3/2}} \]

(Used $\frac{\hat{z}}{z^2} = \frac{\hat{z}^2}{z}$ at one point above)