Problem 1. (25 points)

A cylinder of radius $R$ and length $L$ is aligned and centered on the $z$ axis, and extends from $z=0$ to $z=L$. There is no interior charge and no charge on the end caps, but the cylindrical part of its surface carries a charge $\sigma = Az$.

(a) Find the total charge on the cylinder.

(b) Set up and carry out an integral to find the electric potential $V$ at the center of the end cap at $z=0$.

Hint: You might, or might not, need one of these integrals:

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln(u + \sqrt{u^2 + a^2})$$

$$\int \frac{udu}{\sqrt{u^2 + a^2}} = \sqrt{u^2 + a^2}$$

\[ | 
\begin{align*}
Q &= \left. \left( \int_{z=0}^{z=L} \int_{r=0}^{r=R} \sigma r \, dr \, dz \right) \right|_{s=R} \\
&= 2\pi R \int_{0}^{L} (Az) \, dz \\
&= \pi RL^2
\end{align*}
\]

\[ | 
\begin{align*}
V(P) &= \frac{1}{4\pi \varepsilon_0} 2\pi R \int_{0}^{L} \frac{\sigma \, dz}{\sqrt{z^2 + R^2}} \\
&= \frac{AR}{2\varepsilon_0} \int_{0}^{L} \frac{z \, dz}{\sqrt{z^2 + R^2}} \\
&= \frac{AR}{2\varepsilon_0} \left( \sqrt{z^2 + R^2} - z \right)_{z=0}^{z=L}
\end{align*}
\]
**Problem 2.** (25 points)

Three equal point charges $q$ are located at $(a,0,0)$, $(0,a,0)$, and $(0,0,a)$. (The parts of this problem can be done independently.)

(a) Find the electric field (magnitude and direction) at the point $(a,a,a)$.

(b) What was the work done to assemble these three charges by bringing them from infinity?

(c) Find the work done (by you) move a new charge $Q$ from $(0,0,0)$ to $(a,a,a)$.

\[
(a) \quad \vec{E} \text{ at } (a,a,a) \text{ from } (a,0,0) \colon \quad \vec{E} = \frac{q}{4\pi \varepsilon_0 a^2} \hat{r} = \frac{q}{a\sqrt{2}}
\]

\[
\vec{E} = \frac{q}{8\pi \varepsilon_0 a^3} \left( \frac{0}{a} \right) = \frac{q}{8\pi \varepsilon_0 a^2} \left( \hat{x} + \hat{y} + \hat{z} \right)
\]

(b) \quad W = \frac{1}{\text{paires \ } 4\pi \varepsilon_0} \quad 3 \text{ pairs, } \vec{r}_{ij} = a\sqrt{2}

\[
W = \frac{3q^2}{4\pi \varepsilon_0 a}
\]

(c) \quad W = Q \left[ V(a,a,a) - V(0,0,0) \right]

\[
V(a,a,a) = \frac{1}{4\pi \varepsilon_0} \left( \frac{1}{12a} + \frac{1}{12a} + \frac{1}{12a} \right) = \frac{3q}{4\pi \varepsilon_0 a} \left( \frac{1}{12} \right)
\]

\[
V(0,0,0) = \frac{3q}{4\pi \varepsilon_0 a} \left( \frac{1}{a} + \frac{1}{a} \right) = \frac{3q}{4\pi \varepsilon_0 a}
\]

\[
W = \frac{3qQ}{4\pi \varepsilon_0 a} \left( \frac{1}{12} - 1 \right)
\]
Problem 3. (25 points)

An infinitely long cylinder of radius $a$ is centered on the $z$ axis and carries a variable internal charge density $\rho(s) = \gamma(3s - 2a)$, where $s$ is the distance from the axis. Using Gauss’s Law, find the electric field, both inside ($s < a$) and outside ($s > a$) the cylinder.

**Inside:**

\[ \int_s E \cdot ds = \frac{1}{\varepsilon_0} Q_{\text{enc}} \]

Left-hand side:

\[ LHS = \int_0^s (2\pi s) \gamma (3s - 2a) \, ds \]

\[ Q_{\text{enc}} = 2\pi L \gamma \left( s^3 - 5s^2 a + 8a^3 \right) \]

\[ E(s) = \frac{\gamma}{\varepsilon_0} s(s-a) \]

So

\[ \int E \cdot ds = \frac{1}{\varepsilon_0} 2\pi L \gamma s(s-a) \]

**Outside:**

\[ Q_{\text{enc}} = \int_0^\infty (2\pi s) \gamma (3s^2 - 2a^2) \, ds = 2\pi \gamma a^2 (s-a) = 0 \]

Cylinder is not neutral! \[ E(s) = 0 \text{ outside} \]
Problem 4. (25 points)

A sphere of radius $R$ carries a spherically symmetric interior charge density $\rho(r)$ such that the electric field is $E(r) = E_0 \hat{r}$, that is, the magnitude of $E$ is constant inside the sphere. (Assume $\sigma = 0$ on the surface of the sphere and $\rho = 0$ outside.)

(a) Find $\rho(r)$ inside the sphere.

(b) Find $E(r)$ outside the sphere.

(c) Find $V(r)$ both inside and outside the sphere, taking $V(\infty) = 0$.

\[ a) \quad \rho = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0 \frac{1}{r^2} \frac{d}{dr} r^2 E_0 = \frac{2 \varepsilon_0 E_0}{r} = \rho \]

\[ b) \quad \text{Outside, } \vec{E}(\vec{r}) = \frac{\rho}{4 \pi \varepsilon_0} \frac{1}{r^2} \hat{r} \]

where $Q =$ charge in sphere

\[ = 4\pi \int_0^R r^2 \rho(r) dr \]

\[ = 4\pi \varepsilon_0 E_0 \int_0^R r^2 dr = 4\pi \varepsilon_0 E_0 R^2 \]

\[ \therefore \vec{E}(\vec{r}) = E_0 \frac{R^2}{r^2} \hat{r} \]

(An important note: This is not $\sigma$ at $R$, so expect $E_{in}(r=R) = E_{out}(r=R)$; yes, both $= E_0$.)

\[ c) \]

\[ r>R: \quad V(r) = -\int_{-\infty}^{r} \vec{E} \cdot d\vec{l} = \int_{r}^{\infty} \vec{E} \cdot d\vec{l} = E_0 R^2 \left( -\frac{1}{r^2} \right) _r^{\infty} \]

\[ V(r) = E_0 \frac{R^2}{r} \]

as expected, same as $V$ for a point charge $Q$.

\[ r=R: \quad V(R) = E_0 R \]

\[ r<R: \quad V(r) = -\int_{-\infty}^{R} \vec{E} \cdot d\vec{l} - \int_{R}^{r} \vec{E} \cdot d\vec{l} \]

\[ = E_0 R + (R-r) E_0 = \frac{(2R-r)E_0}{r} \]