Problem 1. (25 points)

Five equal point charges \( q \) are located at the coordinates \((a, 0, 0), (-a, 0, 0), (0, a, 0), (0, -a, 0), \) and \((0, 0, a)\) as shown.

(a) Find the work done to bring another charge \( Q \) from infinity and place it at the origin.

(b) Find the force on the charge at the top vertex (magnitude and direction).

(c) What was the work done to assemble these five charges by bringing them from infinity?

\( W = 5 \times \frac{9Q}{4\pi \varepsilon_0} \frac{1}{a} = \frac{5Q}{4\pi \varepsilon_0 a} \)

\( F_1 = \frac{g^2}{4\pi \varepsilon_0 \frac{a^2}{a^2}} = \frac{g^2}{4\pi \varepsilon_0 \frac{a^2}{a^2}} = \frac{g^2}{4\pi \varepsilon_0 \frac{a^2}{a^2}} \)

\( F_2 = \frac{g^2}{4\pi \varepsilon_0 \frac{a^2}{a^2}} \quad F_3, F_4 \text{ have same } z \text{ part } \quad (x, y \text{ parts cancel}) \)

\( 10 \text{ pairs: } 4 + 4 = 8 \text{ with } d = a \) \( \Rightarrow \frac{g^2}{4\pi \varepsilon_0 a} \frac{1}{a} \)

\( 2 \text{ with } d = 2a \) \( \Rightarrow 2 \times \frac{g^2}{4\pi \varepsilon_0 a} \frac{1}{2a} \)

\( W = \frac{g^2}{4\pi \varepsilon_0 a} (4 \sqrt{2} + 1) \)

(Factor of \( \frac{1}{2} \) not needed when counting each pair once.)
Problem 2. (25 points)

A sphere carries a spherically symmetric interior charge density

\[ \rho(r) = \rho_0 \left(1 - \frac{r}{R}\right) \]

for \( r < R \), and there is vacuum for \( r > R \). Using Gauss’s Law, find the electric field both inside and outside the sphere.

**INSIDE, \( r < R \):**

\[ \int E^2 \, dV = \frac{1}{\varepsilon_0} Q_{\text{enc}} \]

\[ 4\pi r^2 E_r(r) = \frac{1}{\varepsilon_0} \int_0^r r^2 \rho \, dr \]

\[ \int_0^r r^2 \rho \, dr = \int_0^r \rho_0 r^2 \left(1 - \frac{r}{R}\right) \, dr = \rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)_0^r \]

\[ \therefore E_r = \frac{1}{r^2 \varepsilon_0} \rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) \]

\[ E_r = \frac{\rho_0}{\varepsilon_0} r \left(\frac{1}{3} - \frac{r}{4R}\right) \]

\[ E = E_r \hat{r} \]

**OUTSIDE, \( r > R \):**

\[ 4\pi r^2 E_r = \frac{1}{\varepsilon_0} \int_0^R \rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) \]

\[ E_r = \frac{\rho_0}{\varepsilon_0} \frac{R^3}{r^2} \left(\frac{1}{3} - \frac{1}{4}\right) \]

\[ E_r = \frac{\rho_0 R^3}{12 \varepsilon_0 r^2} \]

\[ E = E_r \hat{r} \]
Problem 3. (25 points)

A line of charge of linear density $\lambda_0$ extends infinitely along the $y$ axis. There is a flat rectangular surface patch centered a distance $a$ above this line charge, of length $l$ (parallel to the line) and width $2a$ (area $2al$). That is, the line charge is at $x = z = 0$ and the patch is defined by $-a < x < a, 0 < y < l, z = a$. Calculate the electric flux passing up through this surface patch.

Hint: One of these integrals might be useful:

$$\int \frac{dx}{(x^2 + a^2)^{1/2}} = \ln(x + \sqrt{x^2 + a^2})$$
$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$
$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

Need $\mathbf{E}$ due to line charge: $\mathbf{E} = \frac{\lambda}{2\pi \varepsilon_0 a} \hat{z}$ (cylindrical coords.)

In diagram, $\hat{z} = \hat{E}, \mathbf{s} = \mathbf{a}$

$\mathbf{E} \cdot \mathbf{d\alpha} = E \cos \theta \, d\alpha$

$$\Phi_E = \int_{y=0}^{l} \int_{x=-a}^{a} \mathbf{E} \cdot \mathbf{d\alpha} = \int_{0}^{l} \left[ \frac{\lambda}{2\pi \varepsilon_0 a \sqrt{x^2 + a^2}} \right] dx$$

$$= \frac{\lambda a}{2\pi \varepsilon_0} \left[ \frac{\pi}{4} - \frac{\pi}{4} \right] = \frac{\lambda a}{4\varepsilon_0}$$

*Or, tricky solution via Gauss’s Law:

$$\Phi_E = \frac{1}{4\varepsilon_0} \Phi_{all \ surfs} = \frac{1}{4\varepsilon_0} \Phi_{cyl} = \frac{\lambda l}{4\varepsilon_0}$$

hah!
Problem 4. (25 points)

A uniform volume charge $\rho$ fills the space between the planes $x = -R$ and $x = R$, except that there is an empty spherical cavity of radius $R$ centered in the middle, as shown at right. Find the electric field $\mathbf{E}$ at the point $A$.

Hint: Consider the charge distribution to be the superposition of a positively charged slab and a negatively charged sphere!

Extra credit (+3 points): What is the work done to move a test charge $q$ from point $A$ to point $B$?