**Problem 1.** (25 points)

An infinitely long cylinder of radius $R$ is oriented with its axis along the $z$ axis, and it carries an interior charge density $\rho(s) = \frac{\rho_0 s}{R}$, where $s$ is the distance from the axis. Find the electric field, both inside ($s < R$) and outside ($s > R$) the cylinder.

**Inside**

$$
(2\pi s) (L) E(s) = \frac{1}{\varepsilon_0} \int \rho \, ds
$$

$$
= \frac{1}{\varepsilon_0} \frac{L}{2\pi} \int_0^s \rho(s') s' \, ds'
$$

$$
= \frac{2\pi L}{\varepsilon_0} \frac{\rho_0}{R} \int_0^s (s')^2 \, ds'
$$

$$
= \frac{2\pi L}{\varepsilon_0} \frac{\rho_0}{R} \frac{s^3}{3}
$$

$$
E(s) = \frac{\rho_0}{3\varepsilon_0 R} s^2
$$

**Outside**

$$
Q_{\text{enc}} = 2\pi L \frac{\rho_0 R^3}{3} = \frac{2}{3} \pi L \rho_0 R^2
$$

$$
E(s) = \frac{\rho_0 R^2}{3\varepsilon_0 s}
$$
Problem 2. (25 points)

Two infinitely long line charges run parallel to the $y$ axis in the $x$-$y$ plane, one with density $+\lambda$ at $x = -d/2$ and another with density $-\lambda$ at $x = +d/2$. A point $P$ lies equidistant from the two lines and a distance $h$ above the $x$-$y$ plane. (The figure shows an end view; the dots at $x = \pm d/2$ are line charges seen end on.)

(a) Find the electric field $E$ (magnitude and direction!) at point $P$.

(b) How much work is done to move a point charge $q_0$ from the origin (midway between the lines) to point $P$?

\[
\begin{align*}
\text{(a)} & \quad \vec{E} \text{ from line charge: } \quad \vec{E} = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{s} \hat{x} = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{s^2} \hat{x} \\
& \quad \text{From } +\lambda: \quad s^2 = \frac{d^2}{4} + h^2 \quad \therefore \quad \vec{E}_{\text{from } +\lambda} = \frac{\lambda}{2\pi \varepsilon_0} \frac{\frac{d}{2} \hat{x} + h \hat{z}}{\left(\frac{d}{2}\right)^2 + h^2} \\
& \quad \text{From } -\lambda: \quad \hat{z} \text{ contrib. charges sign, but } \hat{x} \text{ contrib. does not. Thus} \\
& \quad \vec{E}_{\text{tot at } P} = \frac{\lambda}{2\pi \varepsilon_0} \frac{d \hat{x}}{\left(\frac{d}{2}\right)^2 + h^2} \\
& \quad = \frac{2\lambda d \hat{x}}{\pi \varepsilon_0 (d^2 + 4h^2)}
\end{align*}
\]

\[\text{(b) } \vec{E} \text{ is always along } \hat{x}. \quad V = -\int \vec{E} \cdot d\vec{l}, \quad d\vec{l} \text{ along } \hat{z}. \]

Thus $\vec{E} \cdot d\vec{l} \approx \hat{x} \cdot \hat{z} = 0$, no work is done.

\[W = 0\]
Problem 3. (25 points)

For a system with spherical symmetry, the electric field is known to be \( \mathbf{E} = Ar^2 \hat{r} \), where \( A \) is a constant.

(a) Find the electric potential \( V(r) \), using \( r = 0 \) for your reference point \( O \).

(b) Find the charge density \( \rho(r) \).

Hint: The two parts are intended to be independent; you should not need the answer to (a) in order to get (b).

\[
\begin{align*}
(a) \quad V(r) & = - \int_0^r \mathbf{E} \cdot d\mathbf{r} = - A \int_0^r r^2 \, dr = - \frac{A}{3} r^3 \\
& \quad \therefore V = - \frac{A}{3} r^3 \\

(b) \quad \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \mathbf{P} \\
& \quad \therefore \mathbf{P} = \varepsilon_0 \nabla \cdot \mathbf{E} = \varepsilon_0 \left( \frac{1}{r^2} \frac{d}{dr} r^2 E_r \right) \\
& \quad \therefore \mathbf{P} = \varepsilon_0 \frac{1}{r^2} \frac{d}{dr} (Ar^4) \\
& \quad \therefore \mathbf{P} = \frac{4\varepsilon_0 Ar^3}{r^2} \\
& \quad \therefore \rho(r) = \frac{4\varepsilon_0 Ar}{r^2}
\end{align*}
\]
Problem 4. (25 points)

Consider an annulus (flat circular ring) with inner radius $a$ and outer radius $b$, carrying uniform surface density $\sigma$.

(a) Find the total charge of the annulus.

(b) Find the electric potential at a point on the axis located a distance $z$ from the center of the hole.

(c) How much work is required to bring a charge $Q$ from infinity and place it at the center of the hole?

Hints: $\int \frac{t \, dt}{\sqrt{c^2 + t^2}} = \sqrt{c^2 + t^2}$. Also note that when $c \gg d$, $\sqrt{c^2 + d^2} \approx \frac{c^2 + d^2}{2c}$.

\[ a) \quad Q = \int_a^b \sigma \, ds = 2\pi \int_a^b \sigma \, s \, ds = \pi \sigma \int_a^b s \, ds = \pi \sigma \left( \frac{b^2 - a^2}{2} \right) \]

\[ b) \quad V(z) = \frac{1}{4\pi \varepsilon_0} \int_a^b \frac{\sigma (r') \, dr'}{z} = \frac{1}{4\pi \varepsilon_0} \left[ \int_a^b \frac{s \, ds}{\sqrt{s^2 + z^2}} \right] \]

\[ \int_a^b \frac{s \, ds}{\sqrt{s^2 + z^2}} = \left[ \frac{s^2}{2 \sqrt{s^2 + z^2}} \right]_a^b = \frac{b^2 - a^2}{2 \sqrt{b^2 + z^2}} - \frac{a^2}{2 \sqrt{a^2 + z^2}} \]

\[ \therefore \quad V(z) = \frac{\sigma}{2\varepsilon_0} \left[ -\frac{z^2 + b^2}{\sqrt{z^2 + b^2}} + \frac{z^2 + a^2}{\sqrt{z^2 + a^2}} \right] \]

\[ c) \quad \text{Implicit in the formula } V(r) = \frac{1}{4\pi \varepsilon_0} \int \frac{\sigma (r') \, dr'}{r} \]

\[ \text{is that } \sigma \text{ is at } \infty, \text{ so } V(\infty) = 0. \]

Or, take limit:

\[ \lim_{z \to \infty} \left[ \frac{z^2 + b^2}{\sqrt{z^2 + b^2}} - \frac{z^2 + a^2}{\sqrt{z^2 + a^2}} \right] = \lim_{z \to \infty} \left[ z + \frac{b^2}{2z} - z - \frac{a^2}{2z} \right] \]

\[ = \lim_{z \to \infty} \frac{b^2 - a^2}{2z} = 0 \]

\[ \text{Then } W = \frac{Q}{2\varepsilon_0} \left( b - a \right) = W \]