Problem 1 (15 points)

Two infinite line charges of linear density $\lambda$ are parallel to each other at a distance $d$ above an infinite grounded conducting plate. They are a distance $2d$ apart, as shown in cross section in the figure. The point $P$ lies on the planar surface midway between the line charges as shown.

(a) Find $V$ at $P$.

(b) Find $\mathbf{E}$ (magnitude and direction) at $P$.

(c) Find the induced surface charge density $\sigma$ at $P$.

(a) $P$ is on a grounded conductor, so $V = 0$.

On add contributions from 4 line charges; two are $+\lambda$, two are $-\lambda$, all are equidistant, so $V$ adds to zero.

(b) 
\[
\mathbf{E}(P) = \frac{\lambda}{2\pi\varepsilon_0} \frac{s_1}{s_2} \mathbf{s}_1 = \frac{\lambda}{2\pi\varepsilon_0} \mathbf{s}_2
\]

\[
\mathbf{E}_1 = \frac{\lambda}{2\pi\varepsilon_0} \frac{d^2}{(1/2)^2} = \frac{\lambda}{4\pi\varepsilon_0 d} (x'-y')
\]

\[
\mathbf{E}_2 = \frac{\lambda}{4\pi\varepsilon_0 d} (-x'-y') \quad \mathbf{E}_3 = \frac{\lambda}{4\pi\varepsilon_0 d} (-x'-y') \quad \mathbf{E}_4 = \frac{\lambda}{4\pi\varepsilon_0 d} (x'-y')
\]

Total $E_x = 0$, $E_y = -\frac{\lambda}{\pi\varepsilon_0 d} y$

\[
\mathbf{E} = -\frac{\lambda}{\pi\varepsilon_0 d} \hat{y}
\]

(c) $\sigma = \frac{\sigma}{\varepsilon_0}$

$\sigma = -\frac{\lambda}{\pi d}$
Problem 2 (10 points)

You are given that the potential $V(r, \theta) = V_0 \cos(\theta)$ at an inner radius $r = a$, and it equals $2V_0 \cos(\theta)$ at outer radius $r = 2a$. Also, you know that there is no charge filling the shaded vacuum between these two radii.

(a) Find $V(r, \theta)$ in this shaded region, i.e., for $a < r < 2a$.

(b) Convert your answer from (a) to Cartesian coordinates, and calculate the electric field $\mathbf{E}(r)$ in the shaded region.

Hint: If the answer to (b) is surprisingly simple, you are probably on the right track.

\begin{align*}
(a) \quad & V(r, \theta) = \left( A r + \frac{B}{r^2} \right) P_1(\cos \theta) \quad \text{since} \quad \cos \theta = P_1(\cos \theta) \\
& \quad \text{Match at } r=a: \quad (Aa + \frac{B}{a^2}) = V_0 \\
& \quad \text{Match at } r=2a: \quad (2Aa + \frac{B}{4a^2}) = 2V_0 \\
& \quad \text{Eliminate } A: \quad (-2 + \frac{1}{a^2})B = 0 \\
& \quad B = 0 \Rightarrow V(r, \theta) = \frac{V_0}{a} r \cos \theta \\
(b) \quad & \quad \cos \theta = z \quad V(x, y, z) = V_0 \frac{z}{a} \\
& \quad \mathbf{E} = -\nabla V = -\frac{V_0}{a} \hat{z} \quad \text{UNIFORM E-FIELD!} \\
& \quad \text{We are in 3D spherical coordinates, so } r \cos \theta = z, \text{ not } x!
\end{align*}
Problem 3 (10 points)

A disk of radius $R$ lies in the $x$-$y$ plane with its center on the origin. The disk is covered with a uniform surface charge density $\sigma$, and also has a point charge $q = -\pi R^2 \sigma$ at its center. Find the leading behavior of $V(r, \theta)$ far from this object.

\[ I_0 = Q = (\pi R^2) \sigma + (-\pi R^2 \sigma) = 0 \text{ NEUTRAL!} \]

\[ I_1 = Q_2 = 0 \text{ by symmetry (all at } z = 0) \]

\[ I_2 = \int_0^\infty \int_0^{2\pi} r^2 P^2 (\cos \theta) \sigma \, dr \, d\phi \]

\[ = -\frac{\pi \sigma}{\varepsilon_0} \int_0^R r^3 dr = -\frac{\pi \sigma R^4}{4} \]

\[ \therefore V(r, \theta) = \frac{I_2}{4\pi \varepsilon_0} \frac{1}{r^2} P_2 (\cos \theta) \]

\[ V(r, \theta) = -\frac{\sigma R^4}{16 \varepsilon_0} \frac{1}{r^3} P_2 (\cos \theta) = -\frac{\sigma R^4}{32 \varepsilon_0} \frac{3 \cos^2 \theta - 1}{r^3} \]
Problem 4 (15 points)

A thick spherical shell fills the region \( a < r < b \) and carries a given polarization

\[ \mathbf{P}(r) = \frac{\gamma}{r} \hat{r} \]

where \( \gamma \) is a constant. No free charge is present; vacuum fills the regions \( r < a \) and \( r > b \).

a) Find all bound charges, both of type \( \rho_b \) and \( \sigma_b \).

b) Check overall neutrality.

b) Find \( E(r) \) (the electric field that results from the bound charges) in all three regions: \( r < a \), \( a < r < b \), and \( r > b \).

\[ a) \quad \rho_b = -\hat{r} \cdot \nabla \cdot \mathbf{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 P(r) \right) = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \gamma r \right) = -\frac{\gamma r}{r^2} \]

\[ \sigma_b^{\text{in}} = -\frac{\gamma}{a} \quad \sigma_b^{\text{out}} = \frac{\gamma}{b} \]

\[ b) \quad Q_{\text{tot}} = 4\pi a^2 \left( -\frac{\gamma}{a} \right) + 4\pi b^2 \left( \frac{\gamma}{b} \right) + 4\pi \int_a^b \left( \frac{-\gamma r}{r^2} \right) r^2 dr = 4\pi \gamma \left( -a + b - (b-a) \right) = 0 \]

\[ c) \quad 4\pi r^2 E(r) = \frac{1}{\varepsilon_0} Q_{\text{enc}} = 0 \quad \text{for} \quad r < a \quad \text{or} \quad r > b \quad \Rightarrow \quad E = 0 , \quad r < a \quad \text{or} \quad r > b \]

\[ \text{For} \quad a < r < b , \quad Q_{\text{enc}} = -4\pi \gamma a + 4\pi \int_a^r \left( \frac{-\gamma r}{r^2} \right) r^2 dr = -4\pi \gamma r \]

\[ 4\pi r^2 E(r) = -\frac{4\pi \gamma r}{\varepsilon_0} \quad \Rightarrow \quad E(r) = -\frac{\gamma}{\varepsilon_0 r} \]

Alternate solution:

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = 0 \quad \Rightarrow \quad \mathbf{E} = -\frac{1}{\varepsilon_0} \mathbf{P} = -\frac{\gamma}{\varepsilon_0 r} \]
Problem 5 (15 points)

A cylindrical shell of dielectric constant $\epsilon$ with inner radius $a$ and outer radius $b$ has a line charge of linear density $\lambda$ placed down its center. (Assume the cylinder and line charge have infinite length but calculate for a segment of length $L$; the $L$ should fall out.)

a) Find $D(s)$ in each region ($s < a$, $a < s < b$, $s > b$).

b) Find $P(s)$ in the middle region, $a < s < b$, and express your answer in terms of the dielectric susceptibility $\chi$ of the material.

c) Find the bound charge density $\sigma_b$ on the inner surface of the cylindrical shell.

(a) Gauss’s law for $b$: 
\[ \langle \vec{D}(s) = D(s) \hat{s} \rangle \]
\[ 2\pi s \int D(s) \, ds = Q_{\text{enc}} = \lambda \]
\[ \Rightarrow \langle D(s) \rangle = \frac{\lambda}{2\pi s} \text{ Correct in all three regions!} \]

(b) $P(s) = \frac{\chi}{1+\chi} D(s) \Rightarrow \langle P(s) \rangle = \frac{\chi}{1+\chi} \frac{\lambda}{2\pi s}$

(c) $\sigma_b = \hat{P} \cdot \hat{n}$, $\hat{n} = -\hat{\hat{s}}$, $\sigma_b = -\frac{\chi}{1+\chi} \frac{\lambda}{2\pi a}$

Recall $\hat{n}$ points away from dielectric, so $\hat{n} = -\hat{s}$ here.
Problem 6 (15 points)

An infinite cylinder of radius \( a \) contains a uniform volume charge density \( \rho \), and is moving to the right with velocity \( \mathbf{v} = v \hat{x} \). A point charge \( q \) is located a distance \( s \) from the axis of the cylinder \( (s > a) \), and also moves to the right with the same velocity.

a) Calculate the electric force on charge \( q \) (magnitude and direction).

b) Calculate the magnetic force on charge \( q \) (magnitude and direction).

c) For what value of \( v \) does it appear that these forces should cancel?

\[ a) \quad \lambda_{\text{eff}} = \pi a^2 \rho \]
\[ E(s) = \frac{\lambda_{\text{eff}}}{2\pi \varepsilon_0 s} \hat{s} = \frac{\pi a^2 \rho}{2\pi \varepsilon_0 s} \hat{s} = \frac{a^2 \rho}{2\varepsilon_0 s} \hat{s} \]
\[ \mathbf{F} = q E = \frac{a^2 \rho \varepsilon_0}{2s} \hat{s} \]

\[ b) \quad I_{\text{eff}} = \pi a^2 J = \pi a^2 (\rho v) \]
\[ \mathbf{B}(s) = \frac{\mu_0 I_{\text{eff}}}{2\pi s} \hat{\phi} = \frac{\mu_0 \pi a^2 \rho v}{2\pi s} \hat{\phi} = \frac{\mu_0 a^2 \rho v}{2s} \hat{\phi} \]
\[ \mathbf{F} = q \mathbf{v} \times \mathbf{B} = \frac{\mu_0 a^2 \rho v^2}{2s} \hat{\phi} \times \hat{\phi} \]
\[ \mathbf{F} = -\frac{\mu_0 a^2 \rho v^2}{2s} \hat{\phi} \]

\[ c) \quad \mathbf{F}_{\text{ele}} = -\mathbf{F}_{\text{mag}} \Rightarrow \mu_0 v^2 = \frac{1}{\varepsilon_0} \Rightarrow v = \frac{1}{\mu_0 \varepsilon_0} \]

(Speed of light!)
**Problem 7** (10 points)

An infinite slab fills the region $-a < z < a$ and carries a non-uniform current density $\mathbf{J} = \gamma z^2 \hat{x}$. Use Ampère’s Law to find the magnetic field inside the slab only. You can assume that $\mathbf{B}(z) = -\mathbf{B}(-z)$.

Hint: Path $P$ may be a useful choice for the Ampèrian loop.

\[
\begin{align*}
\mathbf{B}(z) &= B(z) \hat{y} \\
\text{Ampère:} & \\
\oint_P \mathbf{B} \cdot d\mathbf{l} &= \mu_0 I_{\text{enc}} \\
- B(z) \ell + B(-z) \ell &= \mu_0 \ell \int_{-z}^{z} \gamma z^2 \, dz \\
-2B(z) &= \mu_0 \left( \frac{\gamma z^3}{3} \right) \bigg|_{-z}^{z} \\
B(z) &= -\frac{\mu_0 \gamma z^3}{3} \\
\mathbf{B}(z) &= -\frac{\mu_0 \gamma z^3}{3} \hat{y}
\end{align*}
\]
**Problem 8** (10 points)

The strangely shaped wire at right lies flat in the $x$-$y$ plane and carries current $I$ as shown. The path has two semicircular arcs of radii $R_1$ and $R_2$ centered on the origin (shown as the dot), and two straight segments that would pass through the origin if extended.

Find the magnetic field at the origin (magnitude and direction).

**Extra Credit (+3):** Compute the total force on the current loop, assuming that a uniform external magnetic field $\vec{B} = B_0 \hat{z}$ is present.

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**FROM PATH 1:**

\[ \vec{B}_0 = \frac{\mu_0 I}{4\pi} \int \frac{dl \times \hat{z}}{R} \]

\[ n = R_1, \quad \hat{n} \times \hat{z} = \hat{\hat{z}}, \quad dl = \pi R_1 \]

\[ \vec{B}_{10} = \frac{\mu_0 I}{4\pi} I (\pi R_1) \frac{1}{R_1^2} = \frac{\mu_0 I}{4 R_1} \]

Similarly,

\[ \vec{B}_{13} = \frac{\mu_0 I}{4 R_2} \]

\[ \vec{B}_{23} = \vec{B}_{45} = 0 \quad \text{since} \quad \hat{n} \parallel dl \Rightarrow \]

\[ \vec{B} = \frac{\mu_0 I}{4} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

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**Extra Credit**

- $F_x = 0$ by symmetry
- $F_{ly} = I \int (dl \times \vec{B}) \cdot \hat{y} = 0$
- $F_{ty} = F_{4y} = 2 I B_0 (R_2 - R_1)$
- $F_{2y} = -2 I B_0 R_2$

Actually, for any closed loop,

\[ F_{ly} = -I B_0 \oint dl_x = 0 \quad \text{since} \quad \oint dl_x = 0 \]

In fact, $\vec{F}_{tot} = 0$. Wow! But only for uniform $\vec{B}$, and the torque is not zero.