6.4.1 Kepler’s second law

As each planet moves around the sun, a line drawn from the planet to the sun sweeps out equal areas in equal time.

Using cylindrical coordinates, \((r,\phi,z)\), choose the \(z\)-axis that is perpendicular to the motion (e.g., planet orbit). Thus,

\[
\vec{\ell} = \vec{r} \times \vec{p} = m \vec{r} \times \dot{\vec{r}} = m \left( \dot{r} \hat{r} + r \dot{\phi} \hat{\phi} \right) = m r^2 \dot{\phi} \hat{z}
\]

\[
\Rightarrow \quad r^2 \dot{\phi} = \frac{(r^2 d\phi)}{dt} \quad \Rightarrow \quad \frac{dA}{2} = \frac{2 dA}{2} = \frac{\ell}{2m} = \frac{1}{2} r^2 \dot{\phi},
\]

where \(\ell = |\vec{\ell}|\).

6.4.2 Angular momentum of \(N\)-particles

Total angular momentum:

\[
\vec{L} = \sum_{\alpha} \vec{\ell}_\alpha = \sum_{\alpha} \vec{r}_\alpha \times \vec{p}_\alpha
\]

\[
\Rightarrow \quad \dot{\vec{L}} = \sum_{\alpha} \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \sum_{\alpha} \vec{r}_\alpha \times \vec{F}_\alpha
\]

The definition of total angular momentum can be generalized to rigid body with continuous mass distribution by replacing summation with integral:

\[
\vec{L} = \int \vec{r} \times \vec{v} \text{d}m = \int \rho (\vec{r} \times \vec{v}) \text{d}^3r
\]

Here \(\vec{F}_\alpha\) are the net force acting on \(\alpha\)th particle. Thus, it means the rate of change of the total angular momentum is the sum of total torques. This is nice but it is not a easy task to count all the forces. We would like to know whether it is possible to separate the contributions from internal and external forces. Like we discussed before,

\[
\vec{F}_\alpha = \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} + \vec{F}_{\alpha}^{\text{ext}}
\]

Thus,

\[
\dot{\vec{L}} = \sum_{\alpha} \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \sum_{\alpha} \vec{r}_\alpha \times \vec{F}_{\alpha}^{\text{ext}}
\]

As we did before, we can regroup the sum of the internal forces such that the actions and reactions are paired. Then we can apply Newton’s third law,

\[
\sum_{\alpha} \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \sum_{\alpha} \sum_{\beta < \alpha} (\vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \vec{r}_\beta \times \vec{F}_{\beta\alpha}) \quad \vec{F}_{\beta\alpha} = -\vec{F}_{\alpha\beta} \quad \sum_{\alpha} \sum_{\beta < \alpha} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta}
\]

If all the particles can be treated as point mass, the internal forces are along the inter-particle direction, i.e., \(\vec{F}_{\alpha\beta} \parallel \vec{r}_{\alpha\beta}\), where \(\vec{r}_{\alpha\beta} \equiv \vec{r}_\alpha - \vec{r}_\beta\). So \((\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta} = 0\). Therefore,
\[ \dot{\mathbf{L}} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{F}^\text{ext}_\alpha = \sum_\alpha \mathbf{\Gamma}^\text{ext}_\alpha \]

In other words, the rate of change of the total angular momentum only depends on the sum of external torques. If there is no net external torque, the total angular momentum is conserved.

### 6.5 Moment of Inertia

As we learned in general physics, angular momentum is useful for describing rotational motion, especially for rigid body. In particular, for rotation of rigid body about a fixed axis (e.g., z-axis), the angular momentum has a simple expression: \( \mathbf{L}_z = \mathbf{I} \cdot \mathbf{\omega} \), where \( \mathbf{I} \) is the moment of inertia of the rigid body about the axis, and \( \mathbf{\omega} \) is the angular velocity. And it can be shown that \( \mathbf{I} = \sum_\alpha m_\alpha r^2_\alpha \), where \( r_\alpha \) is the distance of \( m_\alpha \) from the axis. The integral form is \( \mathbf{I} = \int dm \int r^2 d^3r = \int d^3r \rho r^2 \).

For rotation about z-axis, \( \vec{v}(\vec{r}) = r_\perp \dot{\phi} \hat{\phi} \), so \( \vec{v}_\perp \hat{z} \). Thus, the \( z \) component is \( (\vec{r} \times \vec{v})_z = r_\perp v_\hat{z} = r^2_\perp \dot{\phi} \hat{z} = r^2_\perp \omega \hat{z} \)

\[ \mathbf{L}_z = \int \rho (\vec{r} \times \vec{v})_z d^3r = \int \rho r^2_\perp \omega d^3r = \left( \int \rho r^2_\perp d^3r \right) \omega = \mathbf{I} \cdot \mathbf{\omega} \]

Examples:

- a uniform cylinder (mass \( M \) and radius \( R \), height \( h \)) rotating about its axis: \( \mathbf{I} = \rho \pi R^2 h \)

\[ \mathbf{I} = \int \rho r^2_\perp d^3r = \int \rho r^2_\perp dr d\phi dz = \rho (2\pi)h \int_0^R r^3 dr = \frac{\rho \pi R^4 h}{2} = \frac{1}{2} MR^2 \]

- A solid cone (see the CM problem) rotating about it axis: \( M = \rho \cdot \frac{1}{3} \pi R^2 h \).

\[ \mathbf{I} = \int \rho r^2_\perp d^3r = \rho \int \int r^2 \cdot r dr d\phi dz = \rho (2\pi) \int_0^h dz \int_0^{Rz/h} dr r^3 \]

\[ = 2\pi \rho \int_0^h dz \frac{1}{4} \left( \frac{Rz}{h} \right)^4 = \frac{\pi \rho R^4 h^5}{2h^4 \cdot 5} = \frac{\pi \rho R^4 h}{10} = \frac{3}{10} MR^2 \]

### 6.6 Angular momentum about the Center of Mass

It turns out, the angular momentum with respect to the CM has a special properties. Let’s define the quantities in CM frame with a prime to differentiate from the original inertial frame. So \( \vec{r}_\alpha' = \vec{r}_\alpha - \vec{R} \) and \( \vec{v}_\alpha' = \vec{v}_\alpha - \vec{V} \). Here \( \vec{R} \) and \( \vec{V} \) are position and velocity of the CM. Thus, \( \vec{p}_\alpha' = m_\alpha (\vec{v}_\alpha - \vec{V}) \).
\[ \vec{L}' = \sum_{\alpha} \vec{r}'_\alpha \times m_\alpha \vec{v}'_\alpha = \sum_{\alpha} (\vec{r}_\alpha - \vec{R}) \times [m_\alpha \cdot (\vec{v}_\alpha - \vec{V})] \]

\[ = \sum_{\alpha} (\vec{r}_\alpha \times m_\alpha \vec{v}_\alpha - m_\alpha \vec{r}_\alpha \times \vec{V} - \vec{R} \times m_\alpha \vec{v}_\alpha + m_\alpha \vec{R} \times \vec{V}) \]

\[ = \sum_{\alpha} (\vec{r}_\alpha \times \vec{p}_\alpha) - \vec{R} \times \vec{P} \]

⇒ \[ \vec{L}' = \vec{L} - \vec{L}_{\text{CM}} \]

Thus, the angular momentum about CM equals to the total angular one in lab frame subtract the angular momentum of the CM. The angular momentum about the CM has a very special property, its rate of change only depends on the external torque about CM, (problem 3.37)

\[ \dot{\vec{L}}' = \sum_{\alpha} \dot{\vec{r}}'_\alpha \times m_\alpha \dot{\vec{v}}'_\alpha = \sum_{\alpha} \dot{\vec{r}}'_\alpha \times m_\alpha (\dot{\vec{v}}_\alpha - \dot{\vec{V}}) \]

\[ = \sum_{\alpha} (\vec{r}'_\alpha \times \vec{F}_\alpha^{\text{ext}}) - \left[ \sum_{\alpha} (m_\alpha \vec{r}'_\alpha) \right] \times \dot{\vec{V}} \]

⇒ \[ \dot{\vec{L}}' = \vec{\Gamma}' \]

This is a very useful result. As we will see later that it is true even when CM is not an inertial frame. Thus, it allows dramatic simplification of many-body or rigid body motion.