4.1 beyond linear resistive force

If we ignore gravity, it is possible to solve the equation of motion since it is effectively a 1D problem for \( f(v) \propto v^n \). After some efforts, one can find that for \( 1 < n < 2 \) the object with initial speed \( v_0 \) will travel finite distance. But for \( n \geq 2 \), the object will travel indefinitely. Of course, it will never happen in reality because they will always be overwhelmed by the linear term \(-bv\) when speed is small enough.

5 Charged particles in uniform magnetic field

For a particle with charge \( q \) moving in a uniform magnetic field \( \vec{B} \), the only force acting on it is the Lorentz force: \( \vec{f} = q \vec{v} \times \vec{B} \). Because \( \vec{f} \perp \vec{v} \) at any moment, the motion would be a circular motion if \( \vec{v} \perp \vec{B} \) (something we learned in general physics).

\[
\vec{v} \times \vec{B} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
v_x & v_y & v_z \\
0 & 0 & B
\end{vmatrix} = (Bv_y, -Bv_x, 0) \Rightarrow \begin{cases}
m\dot{v}_x = qBv_y \\
m\dot{v}_y = -qBv_x \\
m\dot{v}_z = 0
\end{cases} \Rightarrow \begin{cases}
\dot{v}_x = \omega v_y \\
\dot{v}_y = -\omega v_x \\
\dot{v}_z = 0
\end{cases}
\]

Here we can define **cyclotron frequency** \( \omega \equiv \frac{qB}{m} \), which is the characteristic (angular) frequency of motion. For simplicity, let’s assume the \( \vec{B} = (0, 0, B) \) is along the \( z \)-axis. So the equations of motion are,

\[
\begin{align*}
\vec{v} \times \vec{B} &= (Bv_y, -Bv_x, 0) \\
\Rightarrow m\dot{v}_x &= qBv_y \\
m\dot{v}_y &= -qBv_x \\
m\dot{v}_z &= 0
\end{align*}
\]

The motion along \( z \) (i.e., magnetic field) is a constant velocity motion, i.e., \( v_z = \text{constant} \). For motion in the \( xy \)-plane that is perpendicular to magnetic field, there are several ways to solve it.

**Method 1** We can decouple \( x \) motion from \( y \):

\[
\dot{v}_x = \frac{d}{dt}(\omega v_y) = \omega \frac{dv_y}{dt} = \omega \cdot (-\omega v_x) = -\omega^2 v_x
\]

This is an equation of harmonic oscillation (to be discussed in Chapter 5). The solution is \( v_x = v_0 \cos(\omega t - \delta) \). Here \( v \) is the amplitude. It is straightforward to show that \( v_y = -v_0 \sin(\omega t - \delta) \). Combined these two solutions, we can see \( \vec{v}(t) \) is rotating with cyclotron frequency, and its magnitude \( |\vec{v}| = \sqrt{v_x^2 + v_y^2} = v_0 \) is a constant. Thus, it describes a constant-speed circular motion with clockwise rotation. If we analyze the problem using right hand rule of cross product, we will find this is indeed the correct answer!

**Method 2** To avoid invoking second order differential equation, we can use a different trick, complex number \( z = x + yi \), where \( i^2 = -1 \) to represents a 2D position \( \vec{r} = (x, y) \). Thus,

\[
\dot{z} = \dot{x} + i \dot{y} = v_x + v_y i, \quad \text{and} \quad \ddot{z} = \ddot{x} + i \ddot{y} = \dot{v}_x + \dot{v}_y i
\]

\[1\] We will learn that \( \dot{\vec{v}} = \vec{v} \times \vec{\omega} \) describes a circular motion later. And \( \omega = \frac{qB}{m} \) is called cyclotron frequency.
For simplicity, let’s define \( \eta \equiv \dot{z} \). Therefore, the 2 equations of motion can be unified,

\[
\dot{\eta} = \ddot{z} = \dot{v}_x + \dot{v}_y i = \omega v_y - \omega v_x i = -i \omega (v_x + v_y i) = -i \omega \dot{z} = -i \omega \eta
\]

So we obtain a simple first order differential equation (though with complex number):

\[
\Rightarrow \dot{\eta} = -i \omega \eta
\]

The solution is \( \eta = \eta_0 e^{-i \omega t} \), where \( \eta_0 \) is a constant. Note that \( \eta_0 = v_0 e^{i \delta} \) is a complex number, so the solution is

\[
\dot{z}(t) = \eta(t) = v_0 e^{i(\delta - \omega t)}
\]

So,

\[
\begin{align*}
    v_x &= \text{Re}(\eta) = v_0 \cos(\omega t - \delta) \\
    v_y &= \text{Im}(\eta) = -v_0 \sin(\omega t - \delta)
\end{align*}
\]

This result agrees with Here we use Euler’s formula \( e^{i \theta} = \cos \theta + i \sin \theta \). The solution is a circle with radius \( v \).

Integrate the velocity, we will get the solution of position \( z(t) \).

\[
z(t) = \int \dot{z} \, dt = \int A e^{i(\delta - \omega t)} \, dt = z_0 + i \frac{A}{\omega} e^{i(\delta - \omega t)}
\]

This is a circle centered at \( z_0 \). The radius of the circle is: \( r = \frac{A}{\omega} = \frac{mv}{qB} \). This radius is called the cyclotron radius. Combined with the constant speed motion along \( z \)-axis, the motion of a charge particle is a spiral.

### 5.0.1 A brief review of complex number

\( z = x + y i \).

- Real part: \( \text{Re}(z) = x \), Imaginary part: \( \text{Im}(z) = y \);
- Complex conjugate: \( \bar{z} = x - y i \);
- Absolute (modulus or magnitude) of \( z \) is \( r = |z| = \sqrt{x^2 + y^2} \).
- Argument (phase) of \( z \): \( \phi = \arg(x + y i) = \tan^{-1} \left( \frac{y}{x} \right) \) \[\text{if } x > 0 \quad \text{and } \left( \frac{y}{x} \right)^2 \]
- Addition: \( z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \);
- Multiplication: \( z = r e^{i \phi} \Rightarrow z_1 \cdot z_2 = (r_1 r_2) e^{i(\phi_1 + \phi_2)} \);\[\text{There is a subtlety of finding unique value of } \phi \text{ from } (x, y) \text{ values. There is a better method which I will not get into.}\]
6  Momentum and Angular momentum

6.1  Momentum conservation

Recall the definition of momentum: \( \vec{p} = m\vec{v} \). As we discussed in previous lectures, if the net external force is zero, i.e., \( \sum \vec{F}_{\text{ext}} = 0 \), then \( \frac{d\vec{p}}{dt} = 0 \), thus \( \vec{p} = \text{constant} \), or \( \vec{p}_i = \vec{p}_f \), which is called \textbf{momentum conservation}.

The momentum conservation law is also applicable for multi-body systems as we discussed in previous lecture. The total momentum \( \vec{P} = \sum_i \vec{p}_i \) is conserved if the sum of all the external forces is zero, i.e., if \( \sum_i \vec{F}_{\text{ext}} = 0 \), \( \vec{P} = \text{constant} \), or \( \vec{P}_i = \vec{P}_f \). Below is an example of momentum conservation.

6.2  Rockets

A nice example of momentum conservation is \textbf{rocket propulsion}. Without external force, the rocket gains propulsion (push) by ejecting burned fuel toward the opposite direction through its motor. Assuming the rocket is moving at \(+x\) direction and ejecting fuel at exhaust speed of \( v_{\text{ex}} \). At time \( t \), the rocket speed is \( v(t) \), thus the momentum is \( p(t) = mv(t) \). After \( dt \) interval, the rocket speed changes \( dv \) by ejecting fuel of mass \( -dm \) with a exhaust speed of \( v_{\text{ex}} \) with respect to rocket. Here \( dm \) is a negative number. The speed of ejected fuel in the rest frame is \( \vec{v} - v_{\text{ex}} \). So the total momentum of the rocket and ejected fuel is,

\[
\vec{P}(t+dt) = (m+dm)(v+dv) - dm(v-v_{\text{ex}}) = mv + m\,dv + v_{\text{ex}}\,dm
\]

Here we ignore \( dm\,dv \) term because it is a second order term. According to the momentum conservation law, \( \vec{P}_i = \vec{P}_f \), so

\[
p(t+dt) = p(t) \Rightarrow mv + m\,dv + v_{\text{ex}}\,dm = mv \Rightarrow m\,dv + v_{\text{ex}}\,dm = 0 \quad \text{or} \quad m\,dv = -v_{\text{ex}}\,dm
\]

Dividing both sides by \( dt \), we can get: \( m(t)\dot{v} = -m v_{\text{ex}} \).

It means the ejection of fuel provide the propulsion, which acts like a force. However, this is different from regular Newton’s 2nd law because the mass term \( m(t) \) is not a constant. Of course, if there is an external force \( \vec{F}_{\text{ext}} \) acting on the rocket (e.g., gravity), it will be added to the right hand side of the above equation. I.e.,

\[
m(t)\dot{v} = -m + F_{\text{ext}}
\]

See \textit{e.g.}, problem 3.8, 3.14, etc.

Now let’s consider the simplest case, no external force. Thus, the problem is reduced to:

\[
dv = -v_{\text{ex}}\frac{dm}{m} \Rightarrow v - v_0 = v_{\text{ex}}\ln\left(\frac{m_0}{m}\right)
\]

You might be surprised by the result because logarithmic function is a very slow increasing function. E.g., even if 90% of the original rocket is fuel, \( i.e., m_0/m = 10 \). The speed gain is only \( 2.3v_{\text{ex}} \), which is not very effective to gain high speed. Thus it is more effective to design multistage rocket. (See problem 3.12)
6.3 Center of Mass

The center of mass (CM) is a very useful concept for studying the motion of many body systems. The position of CM of a $N$-body system is defined as:

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha \quad \text{where} \quad M = \sum_{\alpha=1}^{N} m_\alpha$$

It is easy to find the total momentum of the $N$-body system:

$$\vec{P} = \sum_{\alpha} \vec{p}_\alpha = \sum_{\alpha} m_\alpha \vec{v}_\alpha = \sum_{\alpha} m_\alpha \dot{\vec{r}}_\alpha = M \dot{\vec{R}}$$

This result means the total momentum of the $N$-body system behaves exactly like a point mass $M$ at CM. Thus, $\vec{F}_{ext} = M \ddot{\vec{R}}$.

The concept of CM can be extended to a finite size body with continuous mass distribution, e.g., with position dependent mass density $\rho(\vec{r})$. So $dm = \rho(\vec{r}) dV$, thus,

$$\vec{R} = \frac{1}{M} \int \vec{r} dm = \frac{1}{M} \int \rho(\vec{r}) \vec{r} dV \quad \text{where} \quad M = \int \rho(\vec{r}) dV$$

**Example** – The CM of a solid cone (Fig. 3.4 of Taylor)

A solid cone with uniform density. Its height is $h$ and radius of the base is $R$. Here the tip of cone is on the origin and the axis is along $z$-axis. Because of the symmetry of cone, the CM must lies on the $z$ axis, i.e., $\vec{R} = (0, 0, Z)$. Also, it is more convenient to use cylindrical coordinates $(r, \phi, z)$.

$$Z = \frac{1}{M} \int \rho z dV = \frac{\rho}{M} \int \int z r dr d\phi dz = \frac{\rho}{M} \int_{0}^{h} z \pi R(z)^2 dz$$

For any cross section $0 < z < h$, the radius is $R(z) = \frac{R}{h} z$. So,

$$Z = \frac{\rho}{M} \int_{0}^{h} z \pi \left( \frac{R}{h} z \right)^2 dz = \frac{\rho \pi R^2}{M h} \int_{0}^{h} z^3 dz = \frac{1}{4} \frac{\rho \pi R^2}{M h^4} M = \frac{1}{4} \pi R^2 h \quad \implies \quad Z = \frac{3}{4} h$$

6.4 Angular momentum

The **angular momentum** $\vec{\ell}$ of a moving point mass is defined as $\vec{\ell} = \vec{r} \times \vec{p}$. Strictly speaking, the angular momentum is defined with respect to the origin $O$.

The rate of change of $\vec{\ell}$ is:

$$\dot{\vec{\ell}} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = m \left( \dot{\vec{r}} \times \dot{\vec{r}} \right) + \vec{r} \times \vec{F} = \vec{\Gamma}$$

Here $\vec{\Gamma} \equiv \vec{r} \times \vec{F}$ is the torque acting on the point mass with respect to origin $O$. The torque equation of motion: $\dot{\vec{\ell}} = \vec{\Gamma}$ the rotational version of Newton’s 2nd law. Because the cross-product nature, $\vec{\Gamma} = 0$ if $\vec{r} \parallel \vec{F}$, which is satisfied in **central force problems** where $\vec{\ell}$ is conserved. A famous example is Kepler’s second law on planet motion.