3 Polar coordinates

For every point in 2D space (a plane), we can use either Cartesian coordinates \((x, y)\), or Polar coordinates \((r, \phi)\), where \(r\) is the length of the position vector \(\vec{r}\), and \(\phi\) is the angle of \(\vec{r}\) w.r.t. the \(x\)-axis.

\[
\begin{align*}
x &= r \cos \phi \\
y &= r \sin \phi
\end{align*}
\]

\[
\iff
\begin{align*}
 r &= \sqrt{x^2 + y^2} \\
\phi &= \tan^{-1} \frac{y}{x}
\end{align*}
\]

For every point \(\vec{r}\), we can define \(\hat{r} = \frac{\vec{r}}{r}\), and \(\hat{\phi}\) is perpendicular to \(\hat{r}\). Thus, \(\vec{r} = r\hat{r}\). In general, any vector \(\vec{s}\) can be expressed as \((s_r, s_\phi)\), or \(\vec{s} = s_r \hat{r} + s_\phi \hat{\phi}\).

Note that \(\hat{r}\) and \(\hat{\phi}\) depend on \(\vec{r}\). So they could have time dependence. For motions in polar coordinates, we need to find out velocity \(\vec{v} = \dot{\vec{r}}\) and acceleration \(\vec{a} = \ddot{\vec{r}}\). To do so, we need to find out \(\dot{\hat{r}}\) and \(\dot{\hat{\phi}}\).

\[
\begin{align*}
\dot{r} &= \frac{d\hat{r}}{dt} = \frac{d\phi \cdot 1 \cdot \hat{\phi}}{dt} = \dot{\phi} \hat{\phi} \\
\dot{\phi} &= \frac{d\hat{\phi}}{dt} = \frac{d\phi \cdot 1 \cdot (-\hat{r})}{dt} = -\dot{\phi} \hat{r}
\end{align*}
\]

Armed with this, we can derive expression of velocity \(\vec{v} = \dot{\vec{r}}\) and acceleration \(\vec{a} = \ddot{\vec{r}}\). For velocity,

\[
\vec{v} = \frac{d}{dt}(r\dot{\hat{r}}) = \frac{dr}{dt} \cdot \dot{\hat{r}} + r \cdot \frac{d\dot{\hat{r}}}{dt} = r \dot{\hat{r}} + r \phi \dot{\hat{\phi}}
\]

For acceleration,

\[
\vec{a} = \frac{d}{dt}(r \dot{\hat{r}} + r \phi \dot{\hat{\phi}}) = \dot{r} \dot{\hat{r}} + r \ddot{\hat{r}} + r \dot{\phi} \dot{\hat{\phi}} + r \phi \ddot{\hat{\phi}} + r \phi \dot{\hat{\phi}} = (\dot{r} - r \phi^2) \dot{\hat{r}} + (r \phi + 2 \dot{r} \phi) \dot{\hat{\phi}}
\]

So the Newton’s second law \((\vec{F} = m\vec{a})\) can be expressed as,

\[
\begin{align*}
F_r &= m(\ddot{r} - r \phi^2) \\
F_\phi &= m(r \ddot{\phi} + 2 \dot{r} \phi)
\end{align*}
\]

Although the expression in polar coordinates seems to be more complicate than that in Cartesian one, it is more convenient for describing circular motion.

For example, (Ex. 1.2 on page 30 of Taylor) a mass on a smooth semicircle surface. Since the mass is constrained on the circular surface, there is no radial motion, \(i.e., \dot{r} = 0\text{ and }\ddot{r} = 0\). So there is only one degree of freedom \(\phi\) in this problem. The second law is reduced to,

\[
\begin{align*}
F_r &= -m \cdot r \phi^2 \\
F_\phi &= m \cdot r \ddot{\phi}
\end{align*}
\]

Here \(F_r = mg \cos \phi - N\), and \(F_\phi = -mg \sin \phi\) if we define origin \(O\) is at the center of semicircle, and vertical downward is the \(x\)-axis \((i.e., \phi = 0)\). Then the equations of motion are reduced to

\[
\begin{align*}
mg \cos \phi - N &= -m \cdot r \phi^2 \\
-mg \sin \phi &= m \cdot r \ddot{\phi}
\end{align*}
\]
The first one is for determining constraint force $N$, while the second one is the equation of motion for $\phi$. Canceling out $m$ on both sides, we will get,

$$\ddot{\phi} = -\frac{g}{r} \sin \phi$$

Note that the same differential equation can be applied to other similar problems (e.g., pendulum by replacing $r$ with $l$). It is clear $\phi = 0$ is the equilibrium position $\ddot{\phi} = 0$ of this problem. And for small $\phi$, $\sin \phi \approx \phi$, so the equation of motion is reduced to that of a harmonic oscillator,

$$\ddot{\phi} = -\frac{g}{r} \phi$$

The polar coordinates $(r, \phi)$ in 2D can be generalized to cylindrical coordinates $(r, \phi, z)$ in 3D. Later we will also introduce spherical coordinates $(r, \theta, \phi)$.

4 Projectiles

In general physics, we learned projectile problem without considering air resistance. In this case, the equations of motion can be easily solved.

$$\begin{align*}
  x &= v_0 \cos \theta \cdot t \\
  y &= v_0 \sin \theta \cdot t - \frac{1}{2} gt^2
\end{align*}$$

The range of the projectile on a flat surface is determined by setting $y = 0$:

$$R = \frac{2v_0^2 \sin \theta \cos \theta}{g}$$

In reality, there is always resistive force for object traveling in air, which is against the motion $\vec{v}$. So $\vec{f} = -f(v)\vec{v}$. In general, $f(v)$ is a complicate function, and its form could depend on the magnitude of velocity. So here we make the simplest approximation, considering only the linear order of the polynomial expansion of $f(v)$, i.e., $\vec{f} = -b\vec{v}$, where $b$ is a constant depends on the medium (here it is air). The higher order term $cv^2$ has been discussed in Taylor, which we will skip in this course. Thus, the total force acting on a projectile is, $\vec{F} = -b\vec{v} + m\vec{g}$. Using Cartesian coordinates ($x$ horizontal, $y$ upward vertical), we have:

$$\begin{align*}
  F_x &= -bv_x = m\dot{v}_x \\
  F_y &= -mg - bv_y = m\dot{v}_y
\end{align*} \implies \begin{align*}
  \dot{v}_x &= -\frac{b}{m}v_x \\
  \dot{v}_y &= -(g + \frac{b}{m}v_y)
\end{align*}$$

These are first order differential equations, and can be solved by separation of variables. The first one is relatively easy,

$$\Rightarrow \frac{dv_x}{v_x} = -\frac{b}{m} \, dt \quad \Rightarrow \ln(v_x)\bigg|_{v_{x0}}^{v_x} = -\frac{b}{m} t \bigg|_0^t \quad \Rightarrow \ln \frac{v_x}{v_{x0}} = -\frac{b}{m} t \quad \Rightarrow \quad v_x(t) = v_{x0} \cdot e^{-\frac{t}{\tau}}$$

here $\tau \equiv \frac{m}{b}$ is a decay time constant. Similar method can be applied to $v_y$ equation by substitution $u = g + \frac{b}{m}v_y$, so $\dot{u} = \frac{b}{m}\dot{v}_y$, thus the $v_y$ equation is transformed to, $\frac{m}{b}\ddot{u} = -u$, 


which is the same as that of $v_x$, so the solution is:

$$u(t) = u_0 \cdot e^{-\frac{t}{\tau}} \quad \Rightarrow \quad g + \frac{b}{m}v_x(t) = (g + \frac{b}{m}v_{y0}) \cdot e^{-\frac{t}{\tau}} \quad \Rightarrow \quad v_y(t) = -g\tau + (g\tau + v_{y0}) \cdot e^{-\frac{t}{\tau}}$$

If we define $v_{ter} \equiv g\tau = \frac{mg}{b}$, we can rewrite the solution as:

$$v_y(t) = -v_{ter} + (v_{ter} + v_{y0}) \cdot e^{-\frac{t}{\tau}}$$

Once we solve the velocity, we can integrate it to get displacement. For $x$ component,

$$x = \int_0^t v_x(t) \, dt = \int_0^t v_{x0} \cdot e^{-\frac{t}{\tau}} \, dt = v_{x0}\tau \cdot (1 - e^{-\frac{t}{\tau}})$$

As $t \to \infty$, $x \to v_{x0}\tau$, which is finite.