Quantum Mechanics and Atomic Physics

Lecture 14:
Review for Midterm Exam

http://www.physics.rutgers.edu/ugrad/361

Prof. Eva Halkiadakis
Midterm Exam Info

  - During regular class time (1:40-3pm) and room
  - chapters 1-5, homeworks 1-6 and related lectures
  - it will be closed book but you are allowed to bring one equation sheet
    - one 8.5" x 11" sheet of paper with formulas and notes to consult during the exam. You may write on both sides of this cheat sheet.
    - you should also bring a couple of pencils and a scientific calculator.
  - check website for more information.

- Todays class will be a midterm exam review session.
- Slides posted on course webpage.
How to study

- Read chapters 1-5
  - And the examples in the book!
- Read related lectures
- Review all homework problems in homeworks 1-6
- Go over review session material
- Prepare your formula sheet
- Form study groups
Make sure you know how to draw wave functions and probability distributions for known potentials we solved for in class!
Recall useful animations!

- [https://phet.colorado.edu/en/simulation/legacy/quantum-tunneling](https://phet.colorado.edu/en/simulation/legacy/quantum-tunneling)
Example 1: Bohr Theory

- A “protonium” atom is a bound state of a proton and an antiproton. Using Bohr theory, determine the ground state energy of protonium.

- Assume hydrogen is -13.6eV.

\[
\varepsilon = -\frac{m \cdot \frac{Z^2 e^4}{(4\pi \varepsilon_0)^2} \cdot \frac{1}{2 \pi^2}}{\hbar^2} = -13.6 \text{eV} \frac{1}{n^2}
\]

For Hydrogen:

\[
M_{\text{hydrogen}} = \frac{m_e M_p}{m_e + M_p} = \frac{(0.511 \text{ MeV}) (938 \text{ MeV})}{(0.511 \text{ MeV}) + (938 \text{ MeV})}
\]

\[
\approx 0.511 \text{ MeV}
\]

\[
M_{\text{protonium}} = \frac{M_p \cdot M_p}{M_p + M_p} = \frac{M_p^2}{2M_p} = \frac{M_p}{2} = \frac{0.38 \text{ MeV} c^2}{2} = 0.09 \text{ MeV} c^2
\]
\[ E_{\text{protonium}} = \frac{m_{\text{protonium}}}{m_{\text{hydrogen}}} \times E_{\text{hydrogen}} \]

\[ \frac{\mu_p}{\mu_N} = \frac{469}{0.511} = 918 \]

So for ground state \( n=1 \)

\[ E_{\text{protonium}} = 918 \times (-13.6\,\text{eV}) = -12500\,\text{eV} = -12.5\,\text{keV} \]
Example 2: Bohr Theory II

A particle of mass $m$ moves in a circular orbit in a potential $V = V_0 \frac{r}{a}$, where $V_0$ and $a$ are constants, and $r$ is the orbit radius, so the force is $|F(r)| = \frac{V_0}{a}$. Using the Bohr model, find the energy levels in terms of $V_0$, $a$, $m$, $\hbar$, and a quantum number $n$. The final answer must not have $v$ or $r$ in it.
\[ F(r) = \frac{V_0}{a} = \frac{mv^2}{r} \]
\[ \Rightarrow r = \frac{amv^2}{V_0} \]

Bohr model: \( L = mvr = n\hbar \)
\[ \Rightarrow v = \frac{n\hbar}{mr} \]
\[ \Rightarrow v = \frac{n\hbar}{m} \frac{V_0}{amv^2} \]
\[ \Rightarrow v^2 = \frac{n\hbar V_0}{am^2} \]
\[ \Rightarrow v = (\frac{n\hbar V_0}{am^2})^{\frac{1}{3}} \]

And,
\[ V = V_0 \frac{r}{a} = \frac{V_0}{a} \frac{amv^2}{V_0} = mv^2 \]
\[ \Rightarrow V = mv^2 \]

So,
\[ E = KE + PE = \frac{1}{2}mv^2 + mv^2 = \frac{3}{2}mv^2 \]
\[ E = \frac{3}{2}m \left( \frac{n\hbar V_0}{am^2} \right)^{\frac{2}{3}} \]
\[ \Rightarrow E = \frac{3}{2} \left( \frac{n\hbar V_0}{am^{\frac{2}{3}}} \right) \]
General case:

\[ V(r) = V_0 \left( \frac{r}{a} \right)^k \]

\[ F(r) = -\frac{dV}{dr} = -k V_0 \left( \frac{r}{a} \right)^{k-1} \left( \frac{1}{a} \right) \]

And a similar analysis gives:

\[ E = \left( 1 + \frac{k}{2} \right) \left[ \frac{n^2 \hbar^2 (V_0^2)^{1/k}}{kma^2} \right]^\frac{k}{k+2} \]

So, for \( k = 1 \):

\[ E = \frac{3}{2} \left( \frac{n^2 \hbar^2 V_0^2}{ma^2} \right)^{1/3} \]

Which is what we had previously.
Example 3: Infinite Square Well

\[ V(x) = 0 \quad \text{for} \quad 0 < x < L \]
\[ V(x) = \infty \quad \text{for} \quad x < 0 \text{ and } x > L \]

Eigenvector:
\[ \psi(x) = \sqrt{\frac{2}{3L}} \left( 1 - \cos \frac{4\pi x}{L} \right) \]

- Find the expectation value of energy for the hypothetical eigenfunction above.

\[ E = \frac{p^2}{2m} \]
\[ P_x = -i \hbar \frac{d}{dx} \]
\[ E_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \]
\[ \frac{d\psi}{dx} = \sqrt{\frac{2}{3L}} \left( 1 - \cos \frac{4\pi x}{L} \right) \sin \frac{4\pi x}{L} \]
\[ \frac{d^2\psi}{dx^2} = \sqrt{\frac{2}{3L}} \left( 1 - \cos \frac{4\pi x}{L} \right) \cos \frac{4\pi x}{L} \]
\[ \left< E \right> = \int_0^L 4x \, \bar{e}_{0x} \, \bar{e}_x \, dx = \int_0^L \left[ \frac{2}{3L} \left( 1 - \cos \frac{4\pi x}{L} \right) \left( -\frac{x^2}{2m} \right) \right] \frac{2}{3L} \frac{16\pi^2}{L^2} \cos \frac{4\pi x}{L} \, dx \\
= -\frac{2}{3L} \frac{16\pi^2}{2m} \left( \cos \frac{4\pi x}{L} - \cos^2 \frac{4\pi x}{L} \right) \int_0^L dx \\
= -\frac{16\pi^2 \hbar^2}{3L^3 m} \int_0^L \left( \cos \frac{4\pi x}{L} - \frac{1}{2} - \frac{1}{2} \cos \frac{8\pi x}{L} \right) dx \\
= -\frac{16\pi^2 \hbar^2}{3L^3 m} \left[ \frac{L}{4\pi} \sin \frac{4\pi x}{L} - \frac{x}{2} - \frac{L}{16\pi} \sin \frac{8\pi x}{L} \right]_0^L \\
= -\frac{16\pi^2 \hbar^2}{3L^3 m} \left( -\frac{L}{2} \right) = \frac{8\pi^2 \hbar^2}{3mL^2} \]
This mixed state has $\langle E \rangle$ a little bit more than $E$ for $n=2$. We’ll learn later that a measurement of $E$ will never yield $E_2$!
Example 4: Step+Semi-inf. Well

- A beam of particles of mass \( m \) is incident from the left with amplitude \( A \) and energy \( E>V_0 \).
- Find eigenfunctions and solve for all arbitrary constants in terms of \( A \).
Could write $\Psi_2$ as a sum of sine and cosine, but since we know $\Psi_2(L)=0$, we can automatically satisfy it by writing:

\[ \Psi_2 = C \sin k_2 (x-L) \quad \text{where} \quad k_2 = \frac{1}{\lambda} \sqrt{2m(E-E_0)} \]

- For $x \leq 0$:
  \[ \Psi_1 = A e^{ik_1 x} + B e^{-ik_1 x} \]
  \[ k_1 = \frac{\sqrt{2mE}}{\lambda} \]

- Continuous @ $x=0$:
  \[ A + B = -C \sin k_2 L \]

- $\frac{d\Psi}{dx}$ continuous @ $x=0$:
  \[ ik_1 A - ik_1 B = k_2 C \cos k_2 L \]

Divide two equations:
\[ \frac{A+B}{ik_1 (A-B)} = -\frac{1}{k_2} \tan k_2 L \]
\[ k_2 A + k_2 B = -i k_1 A \tan k_2 L + i k_1 B \tan k_2 L \]

\[ \Rightarrow B = \left( \frac{i k_1 \tan k_2 L + k_2}{i k_1 \tan k_2 L - k_2} \right) A \]

Use: \[ A + B = -C \sin k_2 L \]

to solve for \[ C \] :

\[ -C \sin k_2 L - A = \left( \frac{i k_1 \tan k_2 L + k_2}{i k_1 \tan k_2 L - k_2} \right) A \]

\[ C = -\frac{1}{\sin k_2 L} \left[ \frac{i k_1 \tan k_2 L + k_2 + i k_1 \tan k_2 L - k_2}{i k_1 \tan k_2 L - k_2} \right] A \]

\[ = -\frac{1}{\sin k_2 L} \left[ \frac{2i k_1 \tan k_2 L}{i k_1 \tan k_2 L - k_2} \right] A \]

\[ \Rightarrow C = \frac{2i k_1}{k_2 (\cos k_2 L - i k_1 \sin k_2 L)} A \]
• It’s ok to write

\[ \psi_2 = C \sin k_2 x + D \cos k_2 x \]

or as \[ \psi_2 = C e^{i k_2 x} + D e^{-i k_2 x} \]

- Will get same answer as before but C and D will require lots of messy algebra.

An aside: \( \psi_2(L) = 0 \) but \( \frac{d\psi_2}{dx} \) need not be zero @ \( x = L \).
Exercise

- Try this at home: a fun and easy problem is to show that the reflection coefficient is unity:

\[ R = \frac{B^*B}{A^*A} = 1 \]
Example 5: Half Harmonic Oscillator

For full oscillator we derived:

\[ E_n = \hbar \omega (n + \frac{1}{2}) \quad n = 0, 1, 2, 3, \ldots \]

\[ \psi_n(x) = \frac{1}{\sqrt{\sqrt{n} \pi 2^n n!}} H_n(\alpha x) e^{-\frac{\alpha^2 x^2}{2}} \quad \alpha^2 = \sqrt{\frac{m \hbar}{\hbar^2}} = \frac{m \omega}{\hbar} \]

Half oscillator has extra constraint that:

\[ \psi_n(0) = 0 \]

So \( n \) must be odd

And energy eigenstates are:

\[ E_n = \hbar \omega (n + \frac{1}{2}) \quad n = 1, 3, 5, \ldots \]
Question: Will $\psi_n^{\text{half}}(x) = \psi_n(x)$ with $n = 1, 3, 5, ...$?

Not quite because normalization is now from 0 to infinity instead of $-\infty$ to infinity.

So, $\psi_n^{\text{half}}(x) = \sqrt{2} \psi_n(x)$ for $x > 0$

$= 0$ for $x \leq 0$

and $n=1, 3, 5, ...$

For full oscillator the ground state was $n=0$, but what about the half oscillator?
Now, the half oscillator has ground state $n=1$, so:

\[
\psi_{\text{half}} (x) = \sqrt{2} \sqrt{\frac{\alpha}{2 \sqrt{\pi}}} \cdot 2 \alpha x e^{-\alpha^2 x^2/2} \\
= 2 \sqrt{\frac{\alpha^3}{\sqrt{\pi}}} x e^{-\alpha^2 x^2/2}
\]
Example 6

Could we see QM harmonic oscillations using a microscope?

- Typical visible light: \( \lambda \approx 5000 \text{Å} = 5 \times 10^{-7} \text{m} \)
- Consider an object of diameter: \( d = 10^{-6} \text{m} \)
- If the density is that of typical solids, mass will be: \( m = 10^{-15} \text{kg} \)
- Suppose it is oscillating with frequency \( f = 1000 \text{Hz} \) and amplitude \( A = 10^{-5} \text{m} \)
A) find quantum number $n$

\[ \omega = \sqrt{\frac{k}{m}} \implies \hbar = m \omega^2 \]

\[ k = \left(10^{-5} \text{kg} \right) \left(2\pi \cdot 1000 \text{ Hz} \right)^2 = 4 \times 10^{-8} \frac{N}{m} \]

\[ E = \frac{1}{2} k A^2 = \frac{1}{2} \left(4 \times 10^{-8} \frac{N}{m} \right) \left(10^{-5} m \right)^2 \]

\[ = 2 \times 10^{-18} \text{ J} \]

\[ E = \hbar \omega \left(n + \frac{1}{2} \right) \]

\[ \implies n + \frac{1}{2} = \frac{E}{\hbar \omega} \]

\[ n + \frac{1}{2} = \frac{2 \times 10^{-18} \text{ J}}{\left(1.055 \times 10^{-34} \text{ J} \cdot \text{s} \right) \left(2\pi \cdot 1000 \text{ Hz} \right)} = 3 \times 10^{12} \]
B) what would be the ground state energy?

\[ E_0 = \frac{\hbar \omega}{2} \]

\[ E_0 = \frac{1}{2} \times 1055 \times 10^{-34} \text{ J} \cdot \text{s} \times 2\pi \times 1000 \text{Hz} \]

\[ = 3.3 \times 10^{-31} \text{ J} \]

\[ = 2 \times 10^{-12} \text{ eV} \]

For comparison, thermal energy at room temperature is:

\[ \frac{3}{2} k T = 0.025 \text{ eV} \]

So \( E_0 \) couldn’t be observed!
C) what would be the amplitude in the ground state?

\[
E = \frac{1}{2} k A^2 \Rightarrow A = \sqrt{\frac{2E}{k}}
\]

\[
A = \sqrt{\frac{2 \times 3 \times 10^{-39} \text{ J}}{4 \times 10^{-8} \text{ N/m}}} = 4 \times 10^{-12} \text{ m}
\]

But recall visible light:

\[
\lambda = 5000 \text{ Å} \approx 5 \times 10^{-7} \text{ m}
\]
Summary/Announcements

- Good luck on Midterm exam on Wed.
- Homework #6 due today
- No homework due on Oct 29 - next one will be due on Nov 5.