Solution to HW#7

1. Reed: Chapter 6

Problem 6-1

Consider an electron moving in an infinite potential box of dimensions (1.0, 1.5, 1.9) Å. Tabulate all possible energy levels (in eV) for \((n_x, n_y, n_z) = 1\) to \(3\).

If \(a, b,\) and \(c\) are the \(x, y,\) and \(z\) dimensions of the box, then equation (6.1.15) tells us

\[
E = \frac{\hbar^2}{8m} \left[ \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right].
\]

Substituting numerical values and converting to electron volts gives

\[
E = 37.60 \left[ \frac{n_x^2}{2.25} + \frac{n_y^2}{3.61} \right] \text{eV}.
\]

A total of 27 combinations are possible for \((n_x, n_y, n_z) = 1\) to \(3\). The energies are tabulated below, in increasing values:

<table>
<thead>
<tr>
<th>(n_x)</th>
<th>(n_y)</th>
<th>(n_z)</th>
<th>(E) (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>64.7</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>96.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>114.9</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>146.1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>148.1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>177.6</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>198.2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>198.5</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>208.8</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>227.7</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>229.7</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>258.9</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>260.9</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>281.8</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>311.1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>311.3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>342.6</td>
</tr>
</tbody>
</table>
There are no exactly degenerate levels for $n_x$, $n_y$, $n_z \leq 3$, but there are some (e.g., 1,2,2, and 1,1,3) that are close. An energy level diagram is plotted below.
Problem 6-2

Consider an electron trapped in a box of volume 1 cubic centimeter. How many quantum states are available to the electron below $E = 10$ eV? How many states would lie between 10 eV and 10.01 eV?

From equation (6.1.20), the number of states up to energy $E$ is given by

$$N(E) = \frac{8^{3/2}}{6} \pi \frac{V m^{3/2}}{h^3} E^{3/2}\]

$$

$$= \frac{8^{3/2}}{6} \pi \left(10^{-6} \text{ m}^3\right) \left(9.109 \times 10^{-31} \text{ kg}\right)^{3/2} \left(6.262 \times 10^{-34} \text{ J sec}^{-1}\right) (1.602 \times 10^{-18} \text{ J})^{3/2}$$

$$= 7.179 \times 10^{22}.$$

For $E = 10.01$ eV, we find $N(E) = 7.190 \times 10^{22}$, hence the number of states between 10 and 10.01 eV is $1.077 \times 10^{20}$. 
3. Reed: Chapter 6

Problem 6-4

In his development of blackbody theory, Planck assumed that photons in a blackbody cavity were of a continuum of wavelengths as opposed to the discrete wavelengths that the cavity could actually support on account of its finite dimensions. Planck’s expression for the number of photon states between frequencies $\nu$ and $\nu + d\nu$ that a cavity of volume $V$ could support was

$$N(\nu)\,d\nu = \frac{8\pi V}{c^3} \nu^2 \,d\nu.$$ 

Transform this expression into an equivalent wavelength form. Now consider a tiny volume, say $1 \text{ mm}^3$. How many photon states are possible between $\lambda = 5000\text{Å}$ and $\lambda = 5001\text{Å}$, typical visible-light wavelengths? Was Planck justified in his assumption of an essentially continuous distribution of frequencies/wavelengths?

With $\nu = c/\lambda$, $d\nu = (c/\lambda^2)\,d\lambda$; we ignore the negative sign in this derivative as it indicates only that frequency decreases as wavelength increases. Hence

$$N = \frac{8\pi V}{c^3} \nu^2 \,d\nu = \frac{8\pi V}{c^3} \frac{c^2}{\lambda^2} \lambda^2 \,d\lambda = \frac{8\pi V}{\lambda^4} \,d\lambda.$$ 

With the figures given,

$$N = \frac{8\pi (10^{-9} \text{ m}^3)}{(5 \times 10^{-7} \text{ m})^3} (10^{-10} \text{ m}) = 4.02 \times 10^7.$$ 

About 40 million photon states are available per cubic millimeter between 5000 and 5001Å. Planck was entirely justified in his assumption of a continuous distribution.
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Problem 6-8

In Chapter 8 the angular momentum raising and lowering operators \( L_z = L_x \pm tL_y \) will be introduced. From your results in problem 6-7, show that in spherical coordinates these appear as

\[
L_z = \hbar e^{\pm \phi} \left\{ \pm \frac{\partial}{\partial \theta} + t \cot \theta \frac{\partial}{\partial \phi} \right\}.
\]

From problem 6-7

\[
L_x = -\hbar \left( -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right)
\]

and

\[
L_y = -\hbar \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right).
\]

Now,

\[
L_z = L_x \pm tL_y,
\]

\[
= -\hbar \left( -\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \pm \left( \hbar \cot \theta \left( \cos \phi \pm t \sin \phi \right) \frac{\partial}{\partial \phi} \right)
\]

\[
= \pm \hbar \left( \cos \phi \pm t \sin \phi \right) \frac{\partial}{\partial \theta} \mp \hbar \cot \theta \left( \sin \phi \pm t \cos \phi \right) \frac{\partial}{\partial \phi}
\]

\[
= \pm \hbar \left( \cos \phi \pm t \sin \phi \right) \frac{\partial}{\partial \theta} \mp \hbar \cot \theta \left( \cos \phi \pm t \sin \phi \right) \frac{\partial}{\partial \phi}.
\]

From Euler's identity, \( \cos \phi \pm t \sin \phi = e^{\pm \phi} \), hence

\[
L_z = \hbar e^{\pm \phi} \left\{ \pm \frac{\partial}{\partial \theta} + t \cot \theta \frac{\partial}{\partial \phi} \right\}.
\]
Problem 6-21

In ordinary vector algebra, the cross-product of a vector with itself is always zero: \( \mathbf{A} \times \mathbf{A} = 0 \). When the vectors are operators, however, things can be different. Using spherical coordinates, show that the angular momentum operator behaves as \( \mathbf{L} \times \mathbf{L} = (i\hbar) \mathbf{L} \).

The angular momentum operator is

\[
\mathbf{L}_{op} = -i\hbar \left\{ \phi \frac{\partial}{\partial \theta} - \theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right\}.
\]

Hence

\[
\mathbf{L}_{op} \times \mathbf{L}_{op} = -\hbar^2 \left\{ \phi \frac{\partial}{\partial \theta} - \theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right\} \times \left\{ \phi \frac{\partial}{\partial \theta} - \theta \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right\}.
\]

Expanding out the cross-product gives

\[
\mathbf{L}_{op} \times \mathbf{L}_{op} = -\hbar^2 \left\{ \phi \times \frac{\partial}{\partial \theta} \left( \phi \frac{\partial}{\partial \theta} \right) - \phi \times \frac{\partial}{\partial \theta} \left( \frac{\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right) - \frac{\theta}{\sin \theta} \times \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \theta} \right) + \frac{\theta}{\sin \theta} \times \frac{\partial}{\partial \phi} \left( \frac{\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right) \right\}.
\]

Carrying out the cross-products:

\[
\frac{\mathbf{L}_{op} \times \mathbf{L}_{op}}{-\hbar^2} = \phi \times \left( \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \theta} + \phi \frac{\partial^2}{\partial \theta^2} \right) - \phi \times \left( \frac{1}{\sin \theta} \frac{\partial \theta}{\partial \phi} \frac{\partial}{\partial \theta} + \frac{\theta}{\sin \theta} \cos \theta \frac{\partial^2}{\partial \phi} + \frac{\theta}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \right)
\]

\[
- \frac{\theta}{\sin \theta} \times \left( \frac{\partial \phi}{\partial \phi} \frac{\partial}{\partial \phi} + \phi \frac{\partial^2}{\partial \phi^2} \right) + \frac{\theta}{\sin \theta} \times \left( \frac{1}{\sin \theta} \frac{\partial \theta}{\partial \phi} \frac{\partial}{\partial \phi} + \frac{\theta}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} \right).
\]

Using the results for the derivatives of spherical-coordinate unit vectors (Section 6.2) and recalling that the cross-product of like unit vectors is always zero, this expression reduces to

\[
\frac{\mathbf{L}_{op} \times \mathbf{L}_{op}}{-\hbar^2} = -\phi \times \left( \frac{-r}{\sin \theta} \frac{\partial}{\partial \phi} - \frac{\theta}{\sin^2 \theta} \cos \theta \frac{\partial}{\partial \phi} + \frac{\theta}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right)
\]

\[
- \frac{\theta}{\sin \theta} \times \left( \left( -\sin \theta r - \cos \theta \theta \right) \frac{\partial}{\partial \theta} + \phi \frac{\partial^2}{\partial \theta \partial \phi} \right) + \frac{\theta}{\sin \theta} \times \left( \frac{\cos \phi}{\sin \theta} \frac{\partial}{\partial \phi} + \frac{\theta}{\sin \theta} \frac{\partial^2}{\partial \phi^2} \right).
\]
\[
\frac{L_{\text{op}} \times L_{\text{op}}}{-\hbar^2} = \frac{\theta}{\sin \theta} \frac{\partial}{\partial \phi} - r \frac{\cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \phi} + r \frac{\partial^2}{\sin \theta \partial \theta \partial \phi} - \phi \frac{\partial}{\partial \theta} - \frac{r}{\sin \theta} \frac{\partial^2}{\partial \theta \partial \phi} + \frac{\cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \phi}
\]

Hence

\[
L_{\text{op}} \times L_{\text{op}} = \hbar^2 \left\{ \phi \frac{\partial}{\partial \theta} - \frac{\theta}{\sin \theta} \frac{\partial}{\partial \phi} \right\} = \hbar^2 \left( \frac{L_{\text{op}}}{-\hbar} \right) = + (\hbar) L_{\text{op}}.
\]
6. \[ L = mv \]

The speed of the stone in its orbit is:

\[ v = \frac{2\pi r}{T} = \frac{2\pi (1m)}{1s} = 6.28 \text{ m/s} \]

So,

\[ L = mv = (1 \text{ kg}) (6.28 \text{ m/s}) (1m) = 6.28 \text{ kg m}^2/\text{s} \]

Angular momentum is quantized:

\[ L = \sqrt{\ell (\ell + 1)} \hbar \]

Since we are dealing with a stone—macroscopic scale—then

\[ L \approx \ell \hbar \]  
and \( \ell \) is expected to be large.

So,

\[ \ell = \frac{L}{\hbar} = \frac{6.28 \text{ kg m}^2/\text{s}}{1.055 \times 10^{-34} \text{ kg m}^2/\text{s}} \]

\[ = 5.96 \times 10^{34} \]

Again, we see macroscopic objects are described by enormous quantum numbers. Quantization is not evident on this scale.
(a) \( l = 3 \)

\[
L = \sqrt{l(l+1)} \ h = \sqrt{3(3+1)} \ h = \sqrt{12} \ h
\]

The allowed values of \( l_3 \) are

\[
l_3 = m_e \ h\]

\[|m_e| \leq l\] so

\[m_e = 0, \pm 1, \pm 2, \pm 3\]

\[
\Rightarrow \ l_3 = -3 \ h, -2 \ h, -1 \ h, 0 \ h, 2 \ h, 3 \ h
\]

The allowed values of \( \Theta \) are:

\[
\cos \Theta = \frac{l_3}{L} = \frac{m_e \ h}{\sqrt{l(l+1)} \ h} = \frac{m_e}{\sqrt{12}} = \frac{m_e}{\sqrt{12}}
\]

So,

\[
\cos \Theta = 0, \pm \frac{1}{\sqrt{12}}, \pm \frac{2}{\sqrt{12}}, \pm \frac{3}{\sqrt{12}}
\]

\[
\Rightarrow \ \Theta = 90^\circ, \pm 73.2^\circ, \pm 54.7^\circ, \pm 30^\circ
\]
(b) For 1 kg stone:

\[ \cos \theta = \frac{L_s}{L} = \frac{me^2}{\sqrt{\hbar^2 + \ell^2}} \approx \frac{me}{\ell \hbar} \]

For minimum angle \( me = \ell \)

So,

\[ \cos \theta = \frac{\ell}{\ell} = 1 \]

\[ \Rightarrow \theta \approx 0 \]

So, for atomic electron minimum angle is:

\[ \theta = 30^\circ \]

and for 1 kg stone minimum angle is:

\[ \theta \approx 0^\circ \]