1. Reed: Chapter 3

Problem 3-15

For the potential well illustrated below, sketch a plausible form for a wavefunction corresponding to energy $E$. 

- Increasing $\lambda$ 
- Increasing amplitude at boundaries 
- Exponential decay
2. Reed: Chapter 3

Problem 3-18

In quantum tunneling, the penetration probability is sensitive to slight changes in the height and/or width of the barrier. Consider an electron with $V(x) - E = 10$ eV incident on a barrier of width 20\AA. By what factor does the penetration probability change if the width is increased to 21\AA?

The penetration probability for a barrier of width $x$ is given by $P = e^{-kx}$, where

$$k = \frac{2\sqrt{2m}}{\hbar} \sqrt{V(x) - E}.$$

If $x$ is changed to $x + \Delta x$, then

$$\frac{P_x}{P_{x+\Delta x}} = e^{k\Delta x}.$$

For an electron with $V(x) - E = 10$ eV,

$$k = \frac{2\sqrt{2(9.1094 \times 10^{-31}\text{kg})}}{(1.0546 \times 10^{-34}\text{J - sec})} \sqrt{10(1.6022 \times 10^{-19}\text{J})} = 3.240 \times 10^{10}\text{ m}^{-1}.$$

For $\Delta x = 10^{-10}$ m,

$$\frac{P_x}{P_{x+\Delta x}} = e^{3.24} = 25.5.$$

A 5% increase in the barrier width leads to a drop in the penetration probability by a factor of over 25.
3. Reed: Chapter 3

Problem 3-21

A potential barrier is defined by $V(x) = V_o (1 - x/A)$ for $0 \leq x \leq A$; $V(x) = 0$ otherwise. A particle of mass $m$ and energy $E (< V_o)$ is incident on this barrier from the left. Derive an expression for the penetration probability. Evaluate your result numerically for an electron incident on such a barrier with $V_o = 5 \text{ eV}$, $E = 2 \text{ eV}$, and $A = 12 \text{ Å}$. Potentials of this form are used to model the spontaneous escape of electrons from metal surfaces subjected to electric fields, a process known as cold emission. The expression for the transmission probability is known as the Fowler-Nordheim formula.

A sketch of the potential is shown below. An particle of energy $E (< V_o)$ will cut the barrier at $x = 0$ and at the upper limit $x_{upper} = A(1-E/V_o)$. 

![Diagram of potential barrier](image)
The penetration probability is given by

\[
\ln P = -\frac{2\sqrt{2m}}{\hbar} \int_0^{x_{\text{open}}} \sqrt{V_0(1 - x/A)} - E \, dx = -\frac{2\sqrt{2mV_0}}{\hbar} \int_0^{x_{\text{open}}} \sqrt{(1 - E/V_0) - x/A} \, dx.
\]

Defining the new variable \( y = x/A \) transforms the integral to

\[
\ln P = -\frac{2A\sqrt{2mV_0}}{\hbar} \int_0^{1 - E/V_0} \sqrt{\alpha - y} \, dy,
\]

where \( \alpha = 1 - E/V_0 \).

Solving the integral gives

\[
\ln P = -\frac{2A\sqrt{2mV_0}}{\hbar} \left[ \frac{2}{3} (\alpha - y)^{3/2} \right]_0^{1 - E/V_0} = -\frac{4A\sqrt{2mV_0}}{3\hbar} (1 - E/V_0)^{3/2}.
\]

Substituting the given values (MKS units) yields

\[
\ln P = -\frac{4\left(1.2 \times 10^{-9}\right)\sqrt{2(9.109 \times 10^{-31})(8.010 \times 10^{-19})}}{3(1.055 \times 10^{-34})(0.6)^{3/2}} \approx -8.515,
\]

or \( P \approx 2.0 \times 10^{-4} \).
4. Reed: Chapter 3

Problem 3-22

A potential barrier is defined as

$$V(x) = \begin{cases} 
(V_o/L^2)x^2 & 0 \leq x \leq L \\
0 & x < 0; x > L. 
\end{cases}$$

A particle of mass $m$ and energy $V_o/2$ is incident on this barrier from $x < 0$. Derive an expression for the penetration probability. Evaluate your expression numerically for an electron striking such a barrier with $L = 10\text{Å}$ and $V_o = 5\text{ eV}$.

The situation is sketched below.

![Diagram of a potential barrier with energy levels and x-axis labels]

The barrier penetration integral is

$$\ln P \sim -\frac{2\sqrt{2m}}{\hbar} \int_{b}^{L} \sqrt{V(x) - E} \, dx.$$ 

With $E = V_o/2$, the lower limit of integration is $b = L / \sqrt{2}$, hence
\[ \ln P \sim -\frac{2\sqrt{2mV_0}}{\hbar L} \int_{L/\sqrt{2}}^{L} \sqrt{x^2 - L^2/2} \, dx. \]

This integral is standard (see Appendix C); the result is

\[ \ln P \sim -\frac{\sqrt{2mV_0} L}{\hbar} \left\{ \frac{1}{\sqrt{2}} - \frac{1}{2} \ln\left(1 + \sqrt{2}\right) \right\} \sim -0.3768 \frac{\sqrt{mV_0} L}{\hbar}. \]

For an electron with \( L = 10\,\text{Å} \) and \( V_0 = 5\,\text{eV} \), \( P \sim 0.0473 \).
Problem 3-23

A sprinter of mass 70 kg running at 5 m/s does not have enough kinetic energy to leap a wall of height 5 meters, even if all of that kinetic energy could be directed into an upward leap. If the wall is 0.2 meters thick, estimate the probability of the sprinter being able to “quantum tunnel” through it rather than attempting to leap over it.

The kinetic energy of the sprinter is

\[ E = \frac{1}{2} mv^2 = \frac{1}{2}(70 \text{ kg})(5 \text{ m/s})^2 = 875 \text{ Joules} \]

The sprinter’s potential energy at the top of the wall would be

\[ V_o = mgh = (70 \text{ kg})(9.8 \text{ m/s}^2)(5 \text{ m}) = 3430 \text{ Joules}, \]

showing that the sprinter does not possess sufficient energy to classically leap the wall. We therefore examine the problem as a quantum tunneling issue with a particle of mass 70 kg and energy \( E = 875 \text{ J} \) incident on a barrier of height 3430 J and thickness 0.2 meters. This easily satisfies the condition for a thick barrier,

\[ \frac{2mL^2(V_o-E)}{\hbar^2} \gg 1, \]

since

\[ \frac{2mL^2(V_o-E)}{\hbar^2} \sim 1.29 \times 10^{72}. \]

The penetration probability for a thick barrier is

\[ T_{THICK} = 16 \left( \frac{E}{V_o} \right) \left( 1 - \frac{E}{V_o} \right) \exp \left( -\frac{2L}{\hbar} \sqrt{2m(V_o-E)} \right). \]

This evaluates as

\[ T \sim 3.04 \exp(-2.268 \times 10^{36}). \]

To put this in a more convenient form, invoke the identity \( \ln(x) = 2.303 \log_{10}(x) \), from which we find \( \log_{10}(T) \sim -0.98 \times 10^{36} \), which we can safely round to \( \sim 10^{-36} \), or \( T \sim 10^{-10^{36}} \).
6. Reed: Chapter 3

Problem 3-28

In an experiment involving electron scattering from a finite rectangular well of depth 4 eV, it is found that electrons of energy 5 eV are completely “transmitted”. What must be the width of the well? At what next higher energy can one expect to again observe $T = 1$?

From equation (3.10.15), resonant scattering occurs at

$$L = \frac{n \pi \hbar}{\sqrt{2 m (E + V_0)}} = \frac{(1) \pi (1.055 \times 10^{-34})}{\sqrt{2(9.11 \times 10^{-31})(1.442 \times 10^{-18})}} = 2.045 \times 10^{-10} \text{ meters},$$

or $L = 2.045 \text{ Å}$.

Resonant scattering occurs next for $n = 2$, hence

$$E = \frac{1}{2m} \left( \frac{n \pi \hbar}{L} \right)^2 - V_0 = 5.126 \times 10^{-18} \text{ Joule} = 32 \text{ eV}.$$