Physics 341: Problem Set #10
due November 12

You are encouraged to work in groups on these problems, but each student must write up the solutions individually. You must also list your collaborators on your solutions, and cite any external sources you used (other than the course notes or textbook).

I will give partial credit for partial answers, but only if you show your work and explain your reasoning. Be careful with units.

1. Recall from PS #8 that the Hernquist model for a spherical mass distribution with total mass M and scale length a has a density and enclosed mass

$$\rho(r) = \frac{Ma}{2\pi r(r+a)^3} \qquad M(r) = \frac{Mr^2}{(r+a)^2}$$

   (a) What is the total potential energy of this mass distribution? *Hint:* The answer will depend only on G, M, and a. After making an appropriate substitution, you should find the following integral useful: $\int_0^\infty x^2/(x+1)^5\,dx = 1/12$.

   (b) If the mass distribution is in equilibrium, what is the total kinetic energy?

2. Some time in the future, the Milky Way and Andromeda galaxies will collide. We cannot study the collision in detail without doing computer simulations. However, we can use general principles like conservation of energy and the virial theorem to get an idea of what the final galaxy will be like long after the collision, once the system has settled back down into equilibrium.

   Work in symbols for most of the problem; plug in numbers only at the end.

   (a) For a finite isothermal sphere with a radius R and circular velocity v (remember, the rotation curve is constant), we derived the total mass and potential energy as

$$M = \frac{v^2 R}{G} \qquad U = -\frac{GM^2}{R}$$

   Express the potential energy U in terms of M and v. Use the virial theorem to find the kinetic energy K and total energy E (again in terms of M and v).

   (b) Suppose you start with two identical isothermal spheres, each with initial mass M_i and initial circular velocity v_i, that are at rest a distance d apart. What is the total energy of this system? *Hint:* Consider the total energy for each one in isolation from part (a), and then the potential energy between the two.

(c) Now imagine the two spheres fall toward each other and merge, and that after some time, they equilibrate and end up as a single isothermal sphere. Use conservation of mass, energy, and the virial theorem, to derive the following (and explain your results in words):

   i. the final mass M_f (in terms of the initial mass M_i),
   ii. the final circular velocity v_f (in terms of v_i, R_i, and d), and
   iii. the final radius R_f (in terms of R_i and d).

(d) Now apply your results to a system like the Milky Way and Andromeda — consider two isothermal spheres with circular velocities of 240 km s^{-1} and radii of 180 kpc, which fall from rest at an initial separation of 770 kpc. What are the mass (in $M_\odot$), circular velocity (in km s^{-1}), and radius (in kpc) of the final galaxy?

3. Use the results we derived in class based on the virial theorem to estimate the mass of the Virgo cluster of galaxies, which has a radial velocity dispersion of $\sigma_z = 666$ km s^{-1} and a radius $R = 1.5$ Mpc. You can assume an isothermal sphere model for the mass distribution and that the measured radial velocity dispersion is based on isotropic orbits of identical galaxies. Would your mass estimate increase, decrease, or stay the same if you used a constant density model?