Physics 341: Problem Set #3
due September 24

You are encouraged to work in groups on these problems, but each student must write up the solutions individually. You must also list your collaborators on your solutions, and cite any external sources you used (other than the course notes or textbook).

I will give partial credit for partial answers, but only if you show your work and explain your reasoning. Be careful with units.

1. In Cartesian coordinates, the magnitude of the velocity vector squared is

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v} = v_x^2 + v_y^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

   Show that in polar coordinates

$$|\vec{v}|^2 = v_r^2 + v_\theta^2,$$

   where $v_r = dr/dt$ and $v_\theta = r(d\theta/dt) = r\omega$. *Hint:* Start with $|\vec{v}|^2 = (dx/dt)^2 + (dy/dt)^2$ and substitute for x and y in terms of r and θ.

2. For this problem you can use my `orbit-ps03` spreadsheet, or you can write your own program. You can download an Excel version of the spreadsheet from
   `http://www.physics.rutgers.edu/ugrad/341/orbit-ps03.xls`
   To use the spreadsheet, you will need to edit the fields that are shaded yellow.

   (a) For this problem we will use natural units for the solar system, measuring lengths in AU and time in years. Use Newton's precise version of Kepler's Third Law to show that $GM_\odot = 4\pi^2$ AU^3 yr^{-2}. This value is fixed in the spreadsheet (cell B3).

   (b) We will compute the orbit of Eris, the infamous "tenth planet" that caused Pluto to be demoted to "dwarf planet" status. Eris (also called 2003 UB_{313} in the textbook) has a semimajor axis $a = 67.959$ AU and eccentricity $e = 0.435$ based on the best current observations. Calculate the orbital period P of Eris (in years).

   (c) We can choose the initial conditions to have time $t = 0$ and angle $\theta = 0$, with coordinates centered on the Sun. Let's start at perihelion (closest approach to the Sun), so that $v_r = 0$ at $t = 0$. Determine the remaining initial conditions (ℓ and r_0) you need to reproduce the orbit.

   (d) Plug these initial conditions into the spreadsheet (or your own program) and plot Eris's orbit. You will need to adjust the time step Δt, to cover one full period.

(e) Write down formulas for the perihelion distance, aphelion distance, and semiminor axis in terms of just a and e. Calculate these quantities for Eris and compare them to the values in your spreadsheet orbit. How well do they agree?

(f) The spreadsheet has four additional columns for the tangential velocity (v_θ), the specific kinetic energy ($KE/m = |\vec{v}|^2/2$), the specific potential energy (PE/m), and the specific total energy (E_{total}/m). Write down formulas for these quantities in terms of quantities you have computed already (GM, ℓ, r, v_r, or ω).[1] *Hint:* Make use of your result in Problem #1 above!

(g) Fill in the empty columns in the spreadsheet with the appropriate formulas (or compute these quantities in your own program). Make one plot showing the specific potential, kinetic, and total energy as a function of time. Be sure to label the curves and axes correctly (including units). Verify that the specific total energy is conserved (to within some small numerical error).

(h) Prove that the specific total energy of the orbit is given by

$$\frac{E_{\text{total}}}{m} = -\frac{GM}{2a}$$

Verify that your computed orbit matches this value. *Hint:* Because the total energy is conserved, to derive the formula you can pick any point on the orbit. Try looking at pericenter or apocenter, and take the sum of the specific kinetic and potential energy at one of those points.

[1] Because there are relations between these quantities (for example, $\ell = r^2\omega$), there is more than one right answer. Use any correct formula that is convenient.