\[ \vec{E} = -\vec{\nabla} \phi \]
\[ q = \int_{V} \rho dV \]
\[ \hat{E}(r) = \frac{kQ}{(r-r_1)^2} \]
\[ \phi = \frac{kQ}{r} \]
\[ \hat{E}(r) = \varepsilon \frac{kQ_i}{(r-r_i)^2} \]
\[ \phi = \varepsilon \frac{kQ_i}{r-r_i} \]

\[ \vec{E}(r) = \int_{V} \frac{k \rho(r')} {r'^2} dV' \]
\[ \phi = \frac{1}{\varepsilon} \int_{V} \frac{k \rho(r')} {r'^2} dV' \]

**Gauss's Law**
\[ \int_{V} \vec{E} \cdot d\vec{A} = \Phi = \frac{q}{\varepsilon_0} \]

Useful for symmetric situations

\[ \text{U} = \frac{1}{2} \int_{V} E^2 dV \]

\[ \text{U} = \frac{1}{2} \int_{V} \Phi dV \]

---

*Later in these notes*
Applications

Like 2Q is fixed at +R, -R on y axis

Can you put charges -8 at some \( r \) on x axis, so stable?

Intuition: Small \( |Q| \), \( r \to 0 \)

Large \( |Q| \), \( r \to \infty \), no stable solution

\[
F = \frac{kQq}{r^2}\]

\[
F = \frac{kQq}{(R^2 + r^2)}\]

\[
\frac{kQ^2}{4r^2} = 2 \frac{kQq}{(R^2 + r^2)} \frac{r}{r^2 + q^2} \]

\[
\frac{q}{4r^2} = \frac{2Qr}{(R^2 + r^2)^{3/2}} \]

\[
(R^2 + r^2)^{3/2} = 8Q^2 \frac{r^2}{6} \]

\[
R^2 + r^2 = 4 \left( \frac{Q}{8} \right)^{3/2} r^2 \]

\[
R^2 = \left[ 4 \left( \frac{Q}{8} \right)^{3/2} - 1 \right] r^2 \]

\[
r^2 = \frac{R^2}{\left[ 4 \left( \frac{Q}{8} \right)^{3/2} - 1 \right]} \]

If \( Q = Q \), \( r = R/\sqrt{3} \)

Note: For \( Q > Q \)

denominator < 0

r imaginary

\( \Rightarrow \) No stable solution

- \( Q \)'s \( \to \infty \)

as intuition above

\[
\frac{kQ^2}{4r^2} \to \frac{kQ^2}{4r^2} \cdot \frac{1}{4r^2} \cdot \frac{1}{2} \]

OK
Ring of charge, distance \( r \) from origin, charge \( \rho \), density \( \lambda = \frac{\rho}{2\pi r} \).

What is \( \rho \) at origin?

\[
\rho = \frac{\int dV}{r} = k \lambda \int_0^r \frac{R dr}{r^2 + z^2}
\]

\[
= \pi k \frac{\rho}{2\pi r} \frac{R}{\sqrt{r^2 + z^2}}
\]

All charge same distance from origin, so if we just wrote \( \rho = \frac{k\rho}{r} \frac{R}{\sqrt{r^2 + z^2}} \), we would have been right.

What if we use a cone of charge?

Surface charge \( \sigma \) on the sides. How to do? Add up circles (integrate).

Note area element from integrating along surface, not \( z \). Integrate \( \int \sigma dz \) \( d\theta \) is larger.

\[
\rho = k \sigma 2\pi r \frac{z}{z_0} dz
\]

\[
= k \sigma 2\pi r \frac{z}{z_0} \frac{dz}{\sqrt{r^2 + z^2}}
\]

\[
= k \sigma 2\pi R \frac{z}{z_0} \frac{dz}{\sqrt{r^2 + z^2}}
\]

\[
= k \sigma 2\pi R \frac{z}{z_0} \frac{dz}{\sqrt{r^2 + z^2}}
\]
what is flux through box?
\[ \Phi = \frac{8 \text{enc}}{\varepsilon_0} \]

what is flux through side?
\[ \Phi_{\text{enc. at center}} = \frac{\varepsilon_0}{\varepsilon_0} \ 	ext{for each side.} \]

what if \( \Phi \) is at a corner of the box???

For the 3 sides that touch that corner, \( \Phi \) is in the plane, so \( \vec{E} \parallel dA \)
so \( \Phi = \int \vec{E} \cdot dA = 0 \)

Bad specious reasoning: \( \Phi \) on edge, \( \Phi \) half in, half out, so each side has \( \Phi = \frac{1}{2} \times \varepsilon_0 \frac{8 \text{enc}}{\varepsilon_0} \times \frac{1}{3} = \frac{8 \text{enc}}{3 \varepsilon_0} \)

Wrong!

Correct verbal reasoning: could have 8 cubes meet at the corner, so each get \( \frac{8 \text{enc}}{8 \varepsilon_0} \) because they are symmetric, mirror images. Each of the 3 for sides also are symmetric edge the same \( \Phi \), so

\[ \Phi_{\text{side}} = \frac{1}{3} \times \frac{8 \text{enc}}{\varepsilon_0} = \frac{8 \text{enc}}{24 \varepsilon_0} \]
\[ U = W = \int F \cdot dx = \int_0^\infty \left( k \frac{q_1 q_2}{r^2} \right) \, dt = \frac{k q_1 q_2}{r} \]  

\[ F \text{ and } dx \text{ in opposite directions} \]

\[ \text{No surprise} \]

\[ \text{Why not } \frac{k q_1 q_2}{r} + \frac{k q_2 q_3}{r} ? \text{ one was fixed} \]

\[ \text{For a set of charges, count each pair once (superposition)} \]

\[ U = \frac{1}{2} \sum_{i < j} \frac{k q_i q_j}{r_{ij}} \]

\[ \sum_{i < j} \text{ counts each pair } 2 \times \]

\[ \text{spherical: last time showed} \]

\[ U_{\text{sphere}} = \frac{1}{2} \frac{Q^2}{4\pi \varepsilon_0 R} \]

\[ U_{\text{shell}} = \frac{1}{2} \frac{Q^2}{4\pi \varepsilon_0 R} \]

\[ \Rightarrow \frac{Q}{4\pi \varepsilon_0 R} \]

\[ \text{"r" \"q"} \]

\[ \text{sphere } U = \left( \frac{3}{5} \right) \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{r^2} ? \]

\[ \text{Can we get this from } \frac{1}{2} \int p \varphi \, dV ? \]
Sphere

\[ \Phi_{\text{outside}} = \frac{Q}{4\pi\varepsilon_0 R} \]

\[ \oint S \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \]

\[ E \cdot 4\pi r^2 = \frac{Q}{4\pi R^3} \cdot 4\pi r^3 \cdot \frac{1}{\varepsilon_0} \]

\[ E = \frac{1}{4\pi r^2} \cdot \frac{Q}{\varepsilon_0} \cdot \frac{r^3}{R^3} = \frac{Q}{4\pi\varepsilon_0 R^3} \]

so \[ \Phi_{r \leq R} (r) = \frac{Q}{4\pi\varepsilon_0 R} + \int_{r}^{R} \frac{\frac{Q}{4\pi\varepsilon_0 R^3} dr}{r} \]

\[ = \frac{Q}{4\pi\varepsilon_0 R} + \frac{Q}{4\pi\varepsilon_0 R^3} \left( \frac{r^2}{2} - \frac{R^2}{2} \right) \]

\[ = \frac{Q}{4\pi\varepsilon_0 R} + \frac{Q}{8\pi\varepsilon_0 R} - \frac{Q}{8\pi\varepsilon_0 R^3} r^2 \]

\[ = \frac{Q}{8\pi\varepsilon_0 R^3} (3R^2 - r^2) \]

\[ u = \frac{1}{2} \oint \mathbf{r} \times p \cdot d\mathbf{r} = \frac{1}{2} \cdot 4\pi \sum_{0}^{\infty} \frac{Q}{\frac{4}{3} \pi R^3} \cdot \frac{Q}{8\pi\varepsilon_0 R^3} (3R^2 - r^2) \cdot r^3 dr \]

\[ = \frac{3Q^2}{16\pi\varepsilon_0 R^2} \left[ \frac{3R^2 r^3}{3} - \frac{r^5}{5} \right]_0 \]

\[ = \frac{3Q^2}{16\pi\varepsilon_0 R^6} \left[ R^5 - R^5 \right] \]

\[ = \frac{3Q^2}{16\pi\varepsilon_0 R^6} \cdot \frac{4}{5} R^5 \]

\[ = \frac{3Q^2}{5} \frac{R^5}{16\pi\varepsilon_0 R^6} \]

\[ = \left( \frac{3}{5} \right) \left( \frac{1}{4\pi\varepsilon_0} \right) \frac{Q^2}{R^4} \]

same as before