Lecture 21

Reminder - Exam Wed April 16.
One sheet (both sides) of expressions
Exam parallels homework

Next Monday - review of HW etc.

Topics today:
What sounds good & how to be a composer?
Finish \( \frac{I_T}{I_0} = \frac{1}{1 + FSin^2} \).

\[ F = \frac{4S^2}{(1 - S^2)^2} \]

Or impedance

How wide are these bumps?

\( \frac{I_T}{I} \)

Why peaks go away?

Transmission \( \rightarrow 1 \)

What is impedance?

Power spectrum - example black body radiation

Combination tones - Chapter 8

Important FNC lab

AM - Amplitude Modulation

FM - Frequency Modulation
Birkhoff's Aesthetic Theory

A partial answer may come from the "theory of aesthetic value" propounded by the American mathematician George David Birkhoff (1884–1944). Birkhoff's theory, in a nutshell, says that for a work of art to be pleasing and interesting it should neither be too regular and predictable nor pack too many surprises. Translated to mathematical functions, this might be interpreted as meaning that the power spectrum of the function should behave neither like a boring "brown" noise, with a frequency dependence \( f^{-2} \), nor like an unpredictable white noise, with a frequency dependence \( f^0 \).

In a white-noise process, every value of the process (e.g., the successive frequencies of a melody) is completely independent of its past—it is a total surprise (see Figure 4A). By contrast, in "brown music" (a term derived from Brownian motion), only the increments are independent of the past, giving rise to a rather boring tune (see Figure 4B). Apparently, what most listeners like best, and not only in Bach's time, is music in which the succession of notes is neither too predictable nor too surprising—in other words, a spectrum that varies according to \( f^\alpha \), with the exponent \( \alpha \) between 0 and \( -2 \). As Richard Voss discovered, the exponents found in most music are right near the middle of this range: \( \alpha = -1 \), giving rise to the hyperbolic power law \( f^{-1} \) (see Figure 4C) [VC 78]. Or, as Balthasar van der Pol once said of Bach's music, "It is great because it is inevitable [implying \( \alpha < 0 \)] and yet surprising [\( \alpha > -2 \)]." (I found this quotation in Marc Kac's captivating autobiography, Enigmas of Chance [Kac 85].)

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Figure 4  (A) "White" music produced from independent notes; (B) "brown" music produced from notes with independent increments in frequency; and (C) "pink" music—frequencies and durations of notes are determined by \( 1/f \) (pink noise) [VC 78].

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US INCOME DISTRIBUTION: INEQUALITY FOR ALL
basilar membrane: analysis of sound frequencies

The analysis of sound frequencies by the basilar membrane. (A) The fibres of the basilar membrane become progressively wider and more flexible from the base of the cochlea to the apex. As a result, each area of the basilar membrane vibrates preferentially to a particular sound frequency. (B) High-frequency sound waves cause maximum vibration of the area of the basilar membrane nearest to the base of the cochlea; (C) medium-frequency waves affect the centre of the membrane; (D) and low-frequency waves preferentially stimulate the apex of the basilar membrane. (The locations of cochlear frequencies along the basilar membrane shown are a composite drawn from different sources.)
\[ f(t) = \sum a_n \sin \omega_n t + b_n \cos \omega_n t \]

Power in mode \( n \) is \( a_n^2 + b_n^2 \).

Best example of a power spectrum or energy spectrum.

\[ f_n = n f_0 \]

Most famous intensity spectrum is black body radiation.

\[ n = \text{index of refraction} = n(\lambda) \]

Spectrum of E&M radiation.

Cavity heated to temperature \( T \)

\[ I(\lambda) \propto \lambda^{-1} e^{-\frac{1}{\lambda T}} \]

How much energy, or intensity, or power is in the \( \lambda \) to \( \lambda + \Delta \lambda \) region of the spectrum.

Part of the answer comes from counting the allowed \( \lambda \)'s in the volume - same as architectural acoustic.
The number of states in volume with wavelength $\lambda$ to $\lambda + d\lambda$ is given by $N(\lambda) \sim \text{number of states} \times \text{energy associated with wavelength } \lambda$.

\[ \frac{h^2}{\pi^2 m^2 \lambda^2} = \frac{\hbar^2}{8m} \frac{d}{dx} \left( \frac{1}{\lambda^3} \right) \]

This is done in 3-dimensions.

\[ V = \frac{1}{3} \lambda^3 \cdot \hbar = \text{Planck's constant} \]

See page 569 Eq. 25.2 Architectural Acoustics.

For the number of states (harmonics) from 0 to $f$

\[ N(f) = \frac{4\pi}{3} V \left( \frac{f^2}{f^2} \right)^3 + \text{surface corrections} + 
\text{sound corrections} + 
\text{light corrections} + 
\text{curvature corrections} \]

\[ f = \frac{v}{\lambda} \]

For $\lambda = \frac{v}{f}$

\[ N(\lambda) = \frac{4\pi}{3} V (\lambda)^3 \times \frac{df}{d\lambda} + \frac{4\pi V (\lambda)^2}{\lambda^4} \]

This is important also in statistical physics and astrophysics.
LAST LECTURE (LECTURE 20)

\[ A_0 \rightarrow t \rightarrow A_t \rightarrow r \rightarrow A_{t1} \rightarrow A_{t2} \rightarrow r \rightarrow A_{t3} \]

INCIDENT WAVE

A \rightarrow L \rightarrow B

\[ A_{t1} = A_0 \text{tt} \quad \text{TAKEN PHASE AT B TO BE 0} \]

\[ r = \omega t \quad s = \frac{2l}{v} \]

\[ A_{t2} = A_0 \text{tt} \quad r^2 e^{-i\delta} \]

\[ A_{t3} = A_0 \text{tt} \quad r^4 e^{-2i\delta} \]

\[ \text{LET } A_0 \text{tt} = 1 \]

\[ \frac{I_r}{I_0} = \left| \frac{A_r}{A_0} \right|^2 = \frac{1}{1 + F \sin^2 \frac{s}{2}} \]

\[ F = \frac{4r^2}{(1-r^2)^2 - \left(\frac{2r}{1-r^2}\right)^2} \]

WHEN \[ \sin^2 \frac{s}{2} = 0 \quad \left| \frac{A_r}{A_0} \right|^2 = 1 \]

\[ \text{WHEN } \sin^2 \frac{s}{2} = 1 \quad \frac{s}{2} = \pi, 2\pi, 3\pi, \ldots \]

\[ \text{MINIMUM: } \frac{I_r}{I_0} = \frac{21}{1 + F} \]

\[ \frac{s}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \]

\[ s = \pi, 3\pi, 5\pi, \ldots \]
\[ \frac{I_T}{I_0} = 1 \]

\[ \frac{I_T}{I_0} = \frac{1}{1 + F} \]

\[ \frac{I_T}{I_0} = \frac{1}{2} = \frac{1}{1 + F \sin^2 \frac{\delta}{2}} \]

\[ \Rightarrow F \sin^2 \frac{\delta}{2} = 1 \]

\[ \sin \left( \frac{\delta}{2} \right) = \frac{1}{F} \]

\[ \delta = \frac{\pi}{2} \]

\[ \frac{\partial S}{\partial f} = 2 \pi \delta_0 \quad \Delta f_{1/2} = \text{width in } f\text{-space} \]

\[ \Delta f = \frac{1}{2 \pi t_0} \quad \Delta f_{1/2} = \frac{1}{2 \pi t_0} \]

\[ \Delta f_{1/2} = \frac{\pi}{2} \delta_0 \]

\[ \Delta f_{1/2} \approx \frac{1}{2 \pi t_0} \approx \frac{1}{\pi t_0} \sqrt{F} \approx \frac{1}{\pi t_0} \sqrt{\frac{\pi}{2} \delta_0} \]

\[ \text{for large } F \]
\[
\text{Uncertainty Relation}
\]
\[
\Delta t = \frac{\hbar}{\Delta E} \rightarrow \text{time pulse}
\]
\[
\Delta E \Delta t \sim \frac{\hbar}{\pi}
\]
\[
\Delta E \Delta f \sim \frac{\hbar}{2}\pi
\]
\[
x \Delta t \sim \frac{\hbar}{2}
\]
\[
\Delta E \Delta t \sim \frac{\hbar}{2}
\]

**Note in Musical Instruments**

\[
\eta = \eta(f) \rightarrow 0
\]

Everything is transmitted at high \( f \)
Acoustic Impedance is similar to Resistance

Measure of difficulty or easy to produce flow

Important in brasses, woodwinds, and others

Voltage $V = R \cdot I$

Voltage Current Potential Flow that drives flow

$V \rightarrow$ Pressure $p - p_{atm}$

Labeled $p$ $R_e = \frac{p}{\dot{V} A}$

$I \rightarrow$ Volume Flow $\dot{V}$

$\varepsilon \rightarrow$ Divergence $\nabla \cdot \vec{E} = \frac{\dot{V}}{A}$

A $\rightarrow$ Area

$\dot{V} \rightarrow$ Volume of Fluid in Tube

Particles in Tube Cross $A$

In Time $\dot{V} = (\frac{\dot{V}}{A}) t$

Volume Flow is just $\frac{Volume of Tube}{t}$

Current Flow $J = \frac{\dot{I}}{A}$

$\rho A \frac{\dot{V}}{t}$

$\rho = \frac{P}{\dot{V} A}$

$J = \frac{I}{A} = \frac{\rho A \dot{V}}{t}$

$\frac{\dot{V}}{t}$

Acoustic Impedance $z = \frac{p}{\dot{V}} = \frac{\dot{V} A}{\rho}$
PARTICULAR CASE: SINUSOIDAL VARIATIONS.

CONNECTION OF $Z$, $Z$ WITH INTENSITY.

\[ \Delta p = p = \rho \frac{v_s}{S_m} S_m \sin(2\pi x - \omega t) \]

\[ \text{Speed of particles} \]

\[ S(x,t) = S_m \cos(2\pi x - \omega t) \]

\[ v_p = \left( \frac{\partial S}{\partial t} \right)_x = -\omega S_m \sin(2\pi x - \omega t) \]

\[ T = \left( \frac{1}{2} \right) \rho \frac{v_s}{S_m} (\omega S_m)^2 = \frac{1}{2} \rho v_s \omega^2 S_m^2 \sim \omega^2 \sim f^2 \]

\[ \langle S^2 \rangle = \frac{1}{2} \langle S_m^2 \rangle \]

\[ I = \rho \frac{v_s}{S_m} (\omega S_m) = \frac{1}{2} \rho v_s \omega S_m \]

\[ \rho = \rho \frac{v_s}{S_m} \]

\[ \frac{1}{Z} \rho \frac{1}{v_s} = \frac{\rho}{v_s} = \text{CALLED CHARACTERISTIC IMPEDANCE} \]

\[ Z \]

\[ Z = \frac{\rho}{v_s} \]

\[ Z \text{ IS SMALL WHEN } U \text{ IS LARGE OR } \rho \text{ IS SMALL.} \]

\[ = \frac{\rho}{v_s} \]

\[ Z \text{ IS LARGE WHEN } U \text{ IS SMALL OR } \rho \text{ IS LARGE.} \]

\[ A \]

\[ \text{CHANGE IN CROSS SECTIONAL AREA SUCH AS A CONSTRUCTION CAUSE} \]

\[ Z \text{ TO CHANGE } \rho \text{ LEADS TO REFLECTION OF SOUND WAVES.} \]

\[ \rho = \text{FORCING FUNCTION} \]

\[ v = \text{FORCING FUNCTION} \]

\[ U = \text{FLOW CURRENT} \]

\[ \text{FLOW IN VOLUME SPEED } U \]

\[ \text{CHANGES IN } A \]

\[ R_s = \rho \frac{1}{A} \text{ CAUSES CHANGES IN } R_s \]
Combination tones -
Beats - Lab. - Equal & Unequal Amplitudes

\[ A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \]

If \( \omega_2 = 2 \omega_1 \) then this combination is like Fourier Series with fundamental + 2nd harmonic only.

If \( A_1 = A_2 \) and \( \omega_2 \approx \omega_1 \), then get beats at
\[
\omega = \frac{\omega_2 - \omega_1}{2}
\]

\[ \cos \omega_1 t + \cos \omega_2 t = 2 \cos \frac{\omega_1 + \omega_2}{2} t \cos \frac{\omega_2 - \omega_1}{2} t \]

Use: \( A_1 e^{i \omega_1 t} + A_2 e^{i \omega_2 t} = \overrightarrow{A_1} + \overrightarrow{A_2} \)

[Graphical representation of vectors in complex plane]
Combination Tones.

\[ \vec{A} + \vec{A}_2 = \vec{A}_r \]
\[ \theta_{1,2} = (\omega_2 - \omega_1) t \]

\[ \vec{A}_r \cdot \vec{A}_1 = (\vec{A}_1 + \vec{A}_2) \cdot (\vec{A}_1 + \vec{A}_2) = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\omega_2 - \omega_1) t \]

\[ A_r^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\omega_2 - \omega_1) t \]

LAW OF COSINES.

\[ A_r = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\omega_2 - \omega_1) t} \]

\[ A_r^2 = A_1^2 + A_2^2 - 2A_1 A_2 \cos \phi \]
\[ \Rightarrow A_1^2 + A_2^2 + 2A_1 A_2 \cos \theta_{1,2} \]

\[ A_r \text{ maximum when } (\omega_2 - \omega_1) t = 0, 2\pi, 4\pi \]
\[ \text{vectors are } \uparrow \quad \cos(\omega_2 - \omega_1) t = 1 \]
\[ E' \quad A_r^2 = A_1^2 + A_2^2 + 2A_1 A_2 = (A_1 + A_2)^2 \]
\[ A_r = A_1 + A_2 \]

\[ A_r \text{ minimum when } (\omega_2 - \omega_1) t = \pi, 3\pi, 5\pi \]
\[ \text{vectors are } \downarrow \quad \cos(\omega_2 - \omega_1) t = -1 \]
\[ A_r = |A_r - A_1| \]
COMBINATION TONES

When \((\omega_2 - \omega_1)^2 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\) VECTORS ARE TANGENT TO THE WAVEFORMS.

\[ A_x^2 = A_1^2 + A_2^2 \]

\[ A_y = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t \]

\[ A_y' = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \]

\[ \tan \theta = \frac{A_1 \sin \omega_1 t + A_2 \sin \omega_2 t}{A_1 \cos \omega_1 t + A_2 \cos \omega_2 t} \]

If \(A_1 = A_2\),

\[ \tan \theta = \frac{\sin \omega_1 t + \sin \omega_2 t}{\cos \omega_1 t + \cos \omega_2 t} = \frac{\sin \omega_1 t \cos \omega_2 t + \cos \omega_1 t \cos \omega_2 t}{\cos \omega_1 t \cos \omega_2 t - \sin \omega_1 t \sin \omega_2 t} = \frac{\omega_1 + \omega_2 t}{2} \cos \frac{\omega_2 - \omega_1}{2} t + \frac{\omega_1 - \omega_2 t}{2} \cos \frac{\omega_1 + \omega_2}{2} t \]

\[ \theta = \frac{\omega_1 + \omega_2}{2} t \]
VARIATIONS OF PROBLEM WITH 2 F'S

AM & FM MODULATION AM & E' FM BROADCASTING

F M CARRIER WAVE IS AT HIGH F = fc AND HAS AMPLITUDE Ac \n\[ y = A_c \cos \omega_c t \rightarrow t \]

\[ t = 0 \quad \rightarrow \quad \omega_c t = 2\pi \quad \Rightarrow \quad t = \frac{2\pi}{\omega_c} \]

A SINGER SPEAKS INTO MICROPHONE AT fm

THIS SIGNAL IS CHANGED INTO F M SIGNAL THAT AMPLITUDE MODULATES CARRIER WAVE.

\[ y = A_c \cos 2\pi c t (1 + b_m \cos 2\pi f_m t) \]

IS THE MODULATED WAVE.

\[ y = A_c \cos 2\pi c t + A_c b_m \cos 2\pi f_m t \cos 2\pi f_m t \]

USE \( \cos a \cos b = \frac{1}{2} \cos (a+b) + \frac{1}{2} \cos (a-b) \)

\[ y = A_c \cos 2\pi c t + \frac{1}{2} A_c b_m \cos 2\pi (f_c + f_m) t \]

\[ + \frac{1}{2} A_c b_m \cos 2\pi (f_c - f_m) t \]

FREQUENCIES, MODULATED WAVE \( f_c, f_c + f_m, f_c - f_m \)

\[ \begin{array}{c|c|c}
A & \frac{1}{2} A_c b_m & A_c \\
\hline
f & \frac{1}{2} A_c b_m & f_c \\
\hline
\end{array} \]

SIDE BANDS f_c, f_c + f_m, f_c - f_m

LOWER \( f_c \) \quad UPPER f_c + f_m
$\dot{y} = A\cos(2\pi f t) (1 + \sum b_n \cos(2n\pi f t))$

Small but neglect all products. Or by ETC.

$\dot{y} = A\cos(2\pi f t) (1 + \sum b_n \cos(2n\pi f t))$

Use here Fourier series of the sound wave.

$b_n = \frac{1}{T} \int_0^T f \cos(2n\pi f t) dt$

$\frac{1}{T} \int_0^T f = \frac{1}{T}$
**AM AMPLITUDE MODULATION**

\[ y = A_c \cos 2\pi f_c t \left( 1 + b_m \cos 2\pi f_m t \right) \]

\[ = A_c \cos 2\pi f_c t + \frac{1}{2} A_c b_m \cos 2\pi \left( f_c + f_m \right) t + \frac{1}{2} A_c b_m \cos 2\pi \left( f_c - f_m \right) t \]

\[ \text{AC}[A_c, b_m, f_c, f_m, t_] := A_c \cos[2\pi f_c t] \times (1 + b_m \cos[2\pi f_m t]) \]

\[ \text{Plot}[\{1, -1, \text{AC}[1, 3, 15, 1, t]\}, \{t, 0, 5\}] \]

\[ A_c = 1 \quad A_c \cos 2\pi f_c t \]

\[ f_c = 15 \]

\[ f_m = 1 \]

\[ b_m = 3 < A_c \]

\[ \text{Plot}[\{1, -1, \text{AC}[1.15, 15, 1, t]\}, \{t, 0, 5\}] \]

\[ A_c = 1 \quad A_c \cos 2\pi f_c t \]

\[ f_c = 15 \]

\[ f_m = 1 \]

\[ b_m = 3 < A_c \]

\[ \text{Plot}[\{1, -1, \text{AC}[1.5, 15, 1, t]\}, \{t, 0, 5\}] \]

\[ A_c = 1 \quad A_c \cos 2\pi f_c t \]

\[ f_c = 15 \]

\[ f_m = 1 \]

\[ b_m = A_c \]

\[ b_m > A_c \]

Overmodulated \( y \)

These are not wanted.
\[ AC2 = A_c \cos 2\pi f_c t (1 + b_m \cos 2\pi f_m t) (1 + b_{m2} \cos 2\pi f_{m2} t) \]

\[ \text{GAP} \]

\[ f_c \gg f_m, f_{m2} \]

\[ \pm (f_m \pm f_{m2}) \]

\[ \text{ARE FAR AWAY} \]
FM FREQUENCY MODULATION

CARRIER WAVE

\[ A_c \cos \left( 2\pi f_c t \right) \]

Put inside square bracket [ ] something that modulates carrier wave \( 2\pi f_c t \)

Example

\[ \Psi_{FM} = A_c \cos \left( 2\pi f_c t + b \cos (2\pi f_{FM} t) \right) \]

Old phase + modulating phase

\[ \phi_{FM}(t) = 2\pi f_c t + b \cos (2\pi f_{FM} t) \]

When \( \phi_{FM}(t) = 0, 2\pi, 4\pi, \ldots, 2n\pi, \ldots \)

\[ \Psi_{FM}(t) = A_c \cos \phi_{FM}(t) = +A_c \]

When \( \phi_{FM}(t) = \pi, 3\pi, 5\pi, \ldots, (2n+1)\pi, \ldots \)

\[ \Psi_{FM}(t) = A_c \cos \phi_{FM}(t) = -A_c \]
$\psi(t) = A_c \cos [2\pi f_c t + b \cos (2\pi f_m t)]$

$\phi(t) = 2\pi f_c t + b \cos (2\pi f_m t)$

A to B has bigger slope than $2\pi f_c$  
⇒ smaller time interval to intersect $2\pi f_c$, $(2n+1)\pi$, $(2n+2)\pi$

$\psi_{FM} = A_c - A_c + A_c$...

Peaks are closer in time - higher $f$

B to C has smaller slope than $2\pi f_c$  
⇒ bigger time interval to intersect $2\pi f_c$, $(2n+1)\pi$, $(2n+2)\pi$  

$\psi_{FM} = A_c - A_c - A_c$  

Peaks are further away in time - lower $f$
\[ y(t) = A_c \cos\left[2\pi f_c t + b \cos 2\pi f_m t\right] \]

\[ \text{UNMODULATED} \]
\[ A_c = 1 \]
\[ f_c = 15 \]
\[ b = 0 \]

\[ \text{f MODULATED} \]
\[ A_c = 1 \]
\[ f_c = 15 \]
\[ b = 0.5 \]
\[ f_m = 10 \]

\[ \text{f- MODULATED} \]
\[ A_c = 1 \]
\[ f_c = 15 \]
\[ b = 1 \]
\[ f_m = 10 \]
LECTURE 21

EQUAL TEMPERED SCALE - SEMI-TONE 8' CENT

SEMITONE SPACING INVOLVES \( 2^{\frac{1}{12}} \).

THE OCTAVE IS DIVIDED INTO 12 STEPS
OF EQUAL FREQUENCY RATIO

IF ONE NOTE IS AT \( f_0 \), A SEMITONE
ABOVE IS \( 2^{\frac{1}{12}} f_0 = 1.05946 f_0 \approx 1.06 f_0 \).

NOTE THIS LEADS TO A BIGGER SPACING
AS \( f_0 \) INCREASES - EACH OCTAVE
RAISES THE FREQUENCY BY A FACTOR OF
2.

\[
\begin{align*}
    f_{C_4} &= 261.6 \approx 260 \quad \frac{2}{2} = 520 \\
    f_{C_5} &= 2 f_{C_4} \approx 520 \quad \frac{2}{2} = 1040 \\
    f_{C_6} &= 2 f_{C_5} \approx 1040 \quad \frac{2}{2} = 2080 \\
    f_{C_7} &= 2 f_{C_6} \approx 2080 \quad \frac{2}{2} = 4160
\end{align*}
\]

A MORE REFINED DIVISION IS 1 CENT

\[
2^{\frac{1}{2}} \rightarrow 2^{\frac{1}{1200}} = (2^{\frac{1}{12}})^{\frac{1}{100}}
\]

\[
= \left(1.05946\right)^{\frac{1}{100}} = 1 + \frac{.05946}{100} + \frac{\left(\frac{.05946}{100}\right)^2}{2}
\]

\[
= 1.0005946 \approx 1.0006
\]

1 SEMITONE = 100 CENTS

\[
A = 440 Hz \ \text{SEMITONE ABOVE IS} \approx 440 \times 2^{\frac{1}{12}} \approx 466 Hz \text{ OR } 26.4 \text{ CENT}
\]

\[
A' = 448.2 \text{ SEMITONE ABOVE } A \approx 448.2 \times 2^{\frac{1}{12}} \approx 476.2 Hz \text{ OR } 27.3 \text{ CENT}
\]
**Shepard Tone Illusion**

- Keep amplitude envelope fixed
- Weight given to each octave controls loudness
- Log scale
- Move $f$'s across fixed envelope
- Non-shifting envelope drops off at high and low octaves

Next: Shift frequency components (octaves shown) upward, as frequency goes up past the max, then amplitude drops off and eventually to 0. Meanwhile, a new lowly $f$ appears with increasing amplitude as it shifts upward. One tone fails off at high $f$, new one appears at low $f$.

The resulting impression is a rising pitch that goes on forever. However, it never manages to exceed the high $f$ endpoint.
VIRTUAL PITCH

200 Hz TONE WITH HARMONICS

1. 200 400 600 800 1000...

EAR SENDS FUNDAMENTAL + HARMONICS TO BRAIN

BRAIN NOTES THE SERIES AND IDENTIFIES IT AS 200 Hz f₁

2. 400 Hz

SAME AS 200 - IDENTIFIED AS 400Hz f₁

3. GO BACK TO 1) REMOVE FUNDAMENTAL

MISSING

BRAIN NOTES THE 400, 600, 800, 1000 SEQUENCE

2) IDENTIFIES IT AS 200 Hz SEQUENCE
SECTION 8.8  AURAL HARMONICS

SINGLE TONE AT $f$ IF SUFFICIENTLY LOUD CAN PRODUCE ADDITIONAL PITCHES AT $2f$, $3f$, ... CALLED AURAL HARMONICS.

POWER LAW RESPONSE OF THE EAR

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>$0 - 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sound Pressure</td>
<td>$P = P_0 + P_0 \cdot P$</td>
</tr>
<tr>
<td>$P$ is Sound Pressure</td>
<td></td>
</tr>
</tbody>
</table>

$a_i$'s DETERMINED EXPERIMENTAL.

$p^2$ GENERATES THE SECOND HARMONIC OF THE ORIGINAL PITCH.

$p^3$ GENERATES THE THIRD HARMONIC OF THE ORIGINAL PITCH.

\[
A_i = a \left( e^{i \omega_i t} + e^{-i \omega_i t} \right) + b \left( e^{i \omega_2 t} + e^{-i \omega_2 t} \right)
\]

\[
\chi[A_i^2 - A_f^2 = a_0 + a_1[A_i] + a_2[A_i]^2]
\]

SAME AS INCIDENT
\[ |A_0|^2 = \left( \frac{a}{2} (e^{i\omega_1 t} + e^{-i\omega_1 t}) + \frac{b}{2} (e^{i\omega_2 t} + e^{-i\omega_2 t}) \right)^2 \]

\[ = \left( \frac{a}{2} \right)^2 (e^{i\omega_1 t} + e^{-i\omega_1 t})^2 + \left( \frac{b}{2} \right)^2 (e^{i\omega_2 t} + e^{-i\omega_2 t})^2 \]

\[ + 2 \frac{a b}{2} \frac{1}{2} (e^{i\omega_1 t} + e^{-i\omega_1 t})(e^{i\omega_2 t} + e^{-i\omega_2 t}) \]

\[ = \left( \frac{a}{2} \right)^2 (e^{2i\omega_1 t} + e^{-2i\omega_1 t} + 2)^2 + \left( \frac{b}{2} \right)^2 (e^{2i\omega_2 t} + e^{-2i\omega_2 t} + 2)^2 \]

\[ + 2 \frac{a b}{2} \frac{1}{2} \left( e^{i(\omega_1 + \omega_2) t} + e^{-i(\omega_1 + \omega_2) t} \right) \]

\[ + e^{i(\omega_2 - \omega_1) t} + e^{-i(\omega_2 - \omega_1) t} \]

\[ + e^{i(\omega_2 - \omega_1) t} \]

\[ + e^{i(\omega_1 - \omega_2) t} = \omega_1 > \omega_2 \]

\[ \frac{\omega_1}{2} \quad 2 \omega_1 \]

\[ \omega - \omega_2 \quad \omega_1 + \omega_2 \quad \omega_1 - \omega_2 \]

Cubic term \[ |A_0|^3 = |A_0||A_0|^2 \]

Sidebands will lead to more sidebands.
Non-Linear Systems

Ex. 1

\[ T = mg \cos \theta \]

\[ -T = -l \cdot mg \sin \theta = I \frac{d^2 \theta}{dt^2} \quad I = \frac{I}{2} \]

\[ -g \sin \theta = \frac{d^2 \theta}{dt^2} \]

Small \( \theta \) \sin \theta = 0 \quad -g \theta = \frac{d^2 \theta}{dt^2} \]

\[ \theta = \theta_0 e^{i \omega t} \quad \theta_0 \text{ p.mnt, } \theta_0 \text{ const} \]

\[ -\frac{g}{l} \theta_0 e^{i \omega t} = -\omega^2 \theta_0 e^{i \omega t} \quad \omega = \frac{g}{l} \]

\[ \sin \theta = \theta - \frac{\theta^3}{3!} \]

\[ -\frac{g}{l} \left( \theta - \frac{\theta^3}{3!} \right) = \frac{d^2 \theta}{dt^2} \]

Try \( \theta_0 e^{i \omega t} \)

\[ \frac{g}{2} \left( \theta_0 e^{i \omega t} + \frac{\theta_0^3}{3!} e^{3i \omega t} \right) = -W \frac{d^2 \theta}{dt^2} \]

No solution