ERRORS AND STATISTICS

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INTRODUCTION

It is well recognized that we can never measure any physical magnitude exactly, that is with no error. In discussing errors in individual observations, it is customary to distinguish between systematic errors and chance or random errors.

1. Systematic Errors

Some examples of systematic errors are:

a) Apparatus Errors: meter calibration off; zero off; results influenced by some secondary phenomenon which has been neglected such as contact potentials, etc...; instrument in reproducibility due to slackness and friction of meter bearings, backlash in micrometer or potentiometer measurements.

b) <u>Human Errors:</u> setting the cross hair to one particular side of center; actuating stop watch too soon; ordinary mistakes (these have no place in a well-done experiment and should be completely eliminated).

2. <u>Random or Statistical Errors</u>

These are errors due to the combined effect of a number of causes which tend to produce different results when an experiment is repeated under apparently the same conditions. A large number of such repetitive measurements would tend to converge on the correct result as the number of repetitions increases. Random errors can have either a positive (+) effect or a negative (-) effect. These errors fluctuate: line voltage variations, vibration of instrument supports. Random errors are usually distributed according to a simple law which allows for the use of statistical methods in their determination. Thus, only random errors can be treated mathematically, and this is the only type of error considered in the following analysis.

ACCURACY AND PRECISION

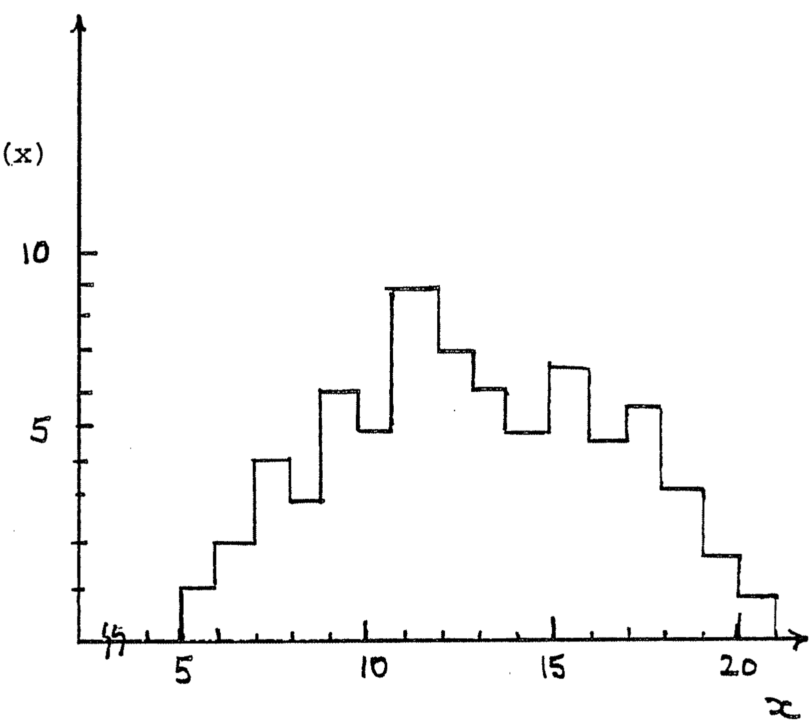
The accuracy of an experiment is a measure of how close the result of the experiment comes to the true value. Therefore, it is a measure of the correctness of the result. The precision of an experiment is a measure of how exactly the result is determined without reference to what that result means. It is also a measure of how reproducible the result is. The absolute precision indicates the magnitude of the uncertainty in the result in the same units as the result. The relative precision indicates the uncertainty in terms of a fraction of the value of the result.

DISTRIBUTIONS IN OBSERVED VALUES OF A GIVEN QUANTITY

If a large number of observations are taken of a quantity x, the number of times N a value of x between x and x + Δx is obtained will vary in accordance to a given distribution best represented by a histogram

Number of N
in Δx or
Probability P(x)

Fig. 1



If the number of readings is very large, the histogram approaches a continuous curve called the <u>distribution curve</u>.

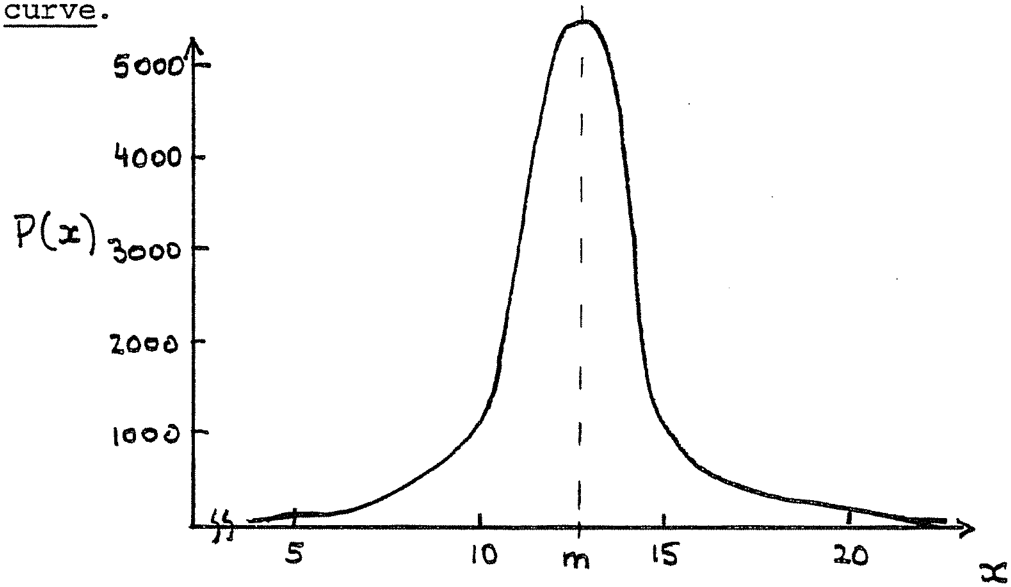


Figure 2

The total area under the curve is equal to the total number of observations. More frequently the curve is normalized by making the area equal to unity. Then $P(x)dx$ is the probability that a given value of x between x and $x + \Delta x$ is measured.

a) The Bi nomial Distribution

The binomial distribution is the fundamental frequency distribution governing random events. The other frequency distributions can be derived from it.

Let p be the probability that an event will occur, $q = 1-p$ the probability that it will not occur, then in a random group of z independent trials, the probability P_x that the eventwill occur x times is given by the term in $p^x q^{z-x}$ in the expansion of $(p+q)^z$. The expansion of $(p+q)^z$ is always equal to unity, and represents the sum of the individual probabilities of observing x=z events, x=z-1 events...., x=0 events.

$$(p+q)^z = p^z + z\,p^{z-1}q + \frac{(z(z-1)}{2!}\,p^{z-2}\,q^2 + \ldots + q^z =$$

$$P_z + P_{z-1} + P_{z-2} + \ldots + P_o = 1 \qquad (1)$$

Any indidivudal term in this binomial expansion can be written as:

$$P_x = \frac{z!}{x!\,(z-x)!}\,p^x\,(1-p)^{z-x} \qquad (2)$$

The best known application of the binomial distribution is to the analysis of tossing coins or throwing dice.
It is usually applied to cases where the variables are discontinuous and integers such as nuclear particle counting rates.

b) <u>The Normal Distribution</u> (Gaussian error curve).

The normal distribution is an analytical approximation to the binomial distribution when z is very large. It is applicable to distributions in which the observed variable is not confined to integer values but can take any value from $-\infty$ to $+\infty$. This distribution is usually applied to a continuous variable. The statistical theory of errors is ordinarily based on the normal distribution

Near the center of the distribution curve, the <u>binomial distribution</u> for large z and constant average value $m = pz$ approaches identity with the <u>normal distribution</u> which states that the probability $dP_x \equiv P(x)\,dx$ that x lies between x and x + dx is

$$dP_x = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-m)^2/2\sigma^2}\, dx \qquad (3)$$

m is the true value of the quantity whose measured values are x, σ is the standard deviation which describes the breadth of the <u>distribution of deviations</u> (x-m) from the mean.

Figure 3 illustrates the general form of the normal distribution for m=100 and σ=10. The curve is normalized so that

$$\int_{\infty}^{\infty} P(x)\,dx \equiv 1$$

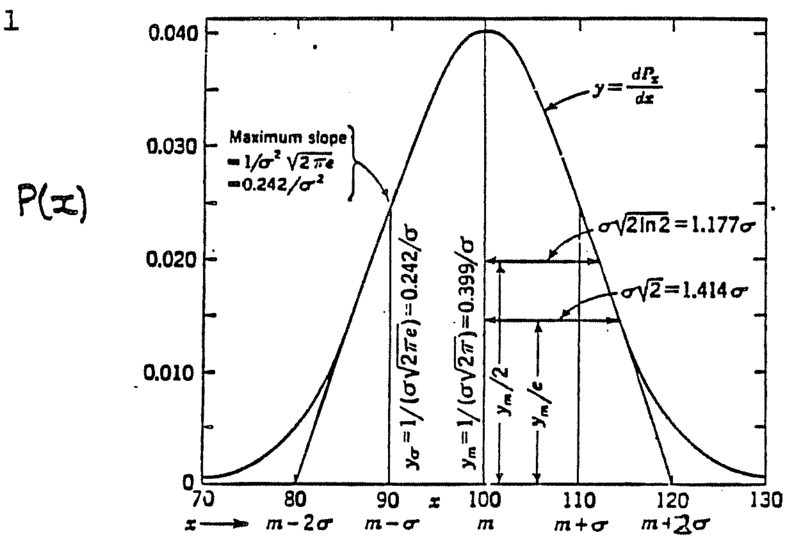


Figure 3. Normal distribution for the special case of a mean value m = 100 and a standard deviation σ=10.

The point of maximum slope $\frac{d^2P(x)}{dx^2} = 0$ falls at $(x-m) = \pm\,\sigma$. At $x = m$, $P(x=m) = \frac{1}{\sigma\sqrt{2\pi}}$ and at $x = m\pm\sigma$,

$$\frac{P(x = m\pm\sigma)}{P(x = m)} \doteq 0.6065.$$

The normal distribution is defined by two parameters, m and σ.

c) <u>The Poisson Distribution</u>

The Poisson distribution <u>describes all random processes whose probability of occurrence is small and constant</u>. It is applicable in the determination of the statistical fluctuations in the number of soldiers kicked yearly by cavalry horses, the disintegration of atomic nuclei, the emission of light quanta by excited atoms and the appearance of cosmic energy bursts. The Poisson distribution applies to substantially all observations made in experimental nuclear physics.

The Poisson distribution can be deduced as a limiting case of the binomial distribution where the probability of occurrence p is very small, $p \ll 1$, while the number of trials z becomes very large $z \gg 1$, while the mean value $m=pz$ remains fixed.

Then if we expand the individual terms in the binomial distribution, and assume $z >> x$

$$\frac{z!}{(z-x)!} = z\ (z-1)\ (z-2)\ldots.\ (z-x+2)\ (z-x+1)\ \rightarrow\ z^{x}$$

$$\lim_{p \rightarrow o}\ (1-p)^{z-x}\quad e^{-p(z-x)} \rightarrow e^{-pz}$$

and thus for $m << z$, and $x << z$

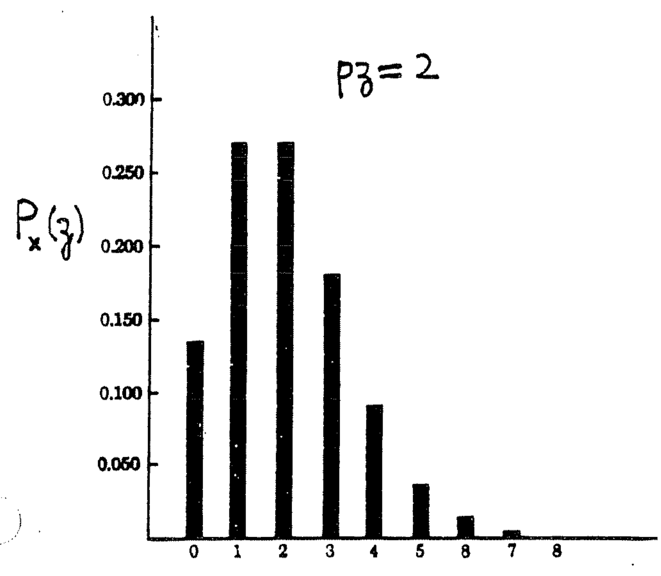
$$P_x = \frac{z!}{x!(z-x)!}\ p^{x}\ (1-p)^{z-x}\ \rightarrow\ \frac{z^{x}p^{x}}{x!}\ e^{-pz} \tag{4}$$

which yield the Poisson distribution

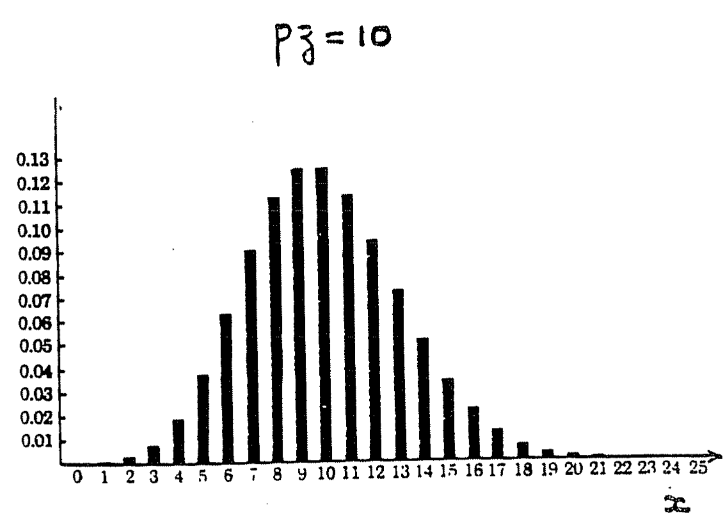
$$P_x = \frac{m^{x}}{x!}\ e^{-m} \qquad \text{and} \qquad \sum_{x=o}^{x=\infty} P_x = 1 \tag{5}$$

Note that the Poisson distribution has only one parameter, m. (For an exact derivation of the Poisson discontribution, see Bevington page 38).

When the number of independent events becomes very large ($z >> 1$) it can be shown that the Poisson distribution approaches the Gaussian form (for the mathematical derivation see Baird, Appendix 1; Young , Appendix C).



Poisson distribution with pz = 2.
Figure 4.



Poisson distribution with pz = 10.

STATISTICAL CHARACTERIZATION OF DATA

A) Mean Value

In a finite series of measurements, we can never find the exact value of the mean $\bar{x}$ which corresponds to an infinite amount of data. Individual measurements will be distributed about the mean according to the particular frequency distribution that applies to the process under study.

The mean value $\bar{x}$ is the best approximation to the true mean m. $\bar{x}$ is given by the arithmetic average of n separate measurements:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \approx m \tag{6}$$

B) Standard Deviation and Variance

a) Definitions

The standard deviation σ is the quantitative expression of the breadth of the statistical fluctuations of the individual readings about the true mean.

For any frequency distribution

$$\sigma^2 \equiv \sum_{x=\infty}^{x=+\infty} (x - m)^2 \; P(x) \tag{7}$$

or in terms of a large series of n measurements of x,

$$\sigma^2 \equiv \frac{1}{n} \sum_{i=1}^{n} (x_i - m)^2 \tag{8}$$

σ^2 is called the variance, σ is occasionally called the root-mean square error.

In a normal distribution, 32% of a large series of individual observations deviate from the mean by more than $\pm\,\sigma$ and 68% of the individual observations lie within the band $\bar{x} \pm \sigma$.

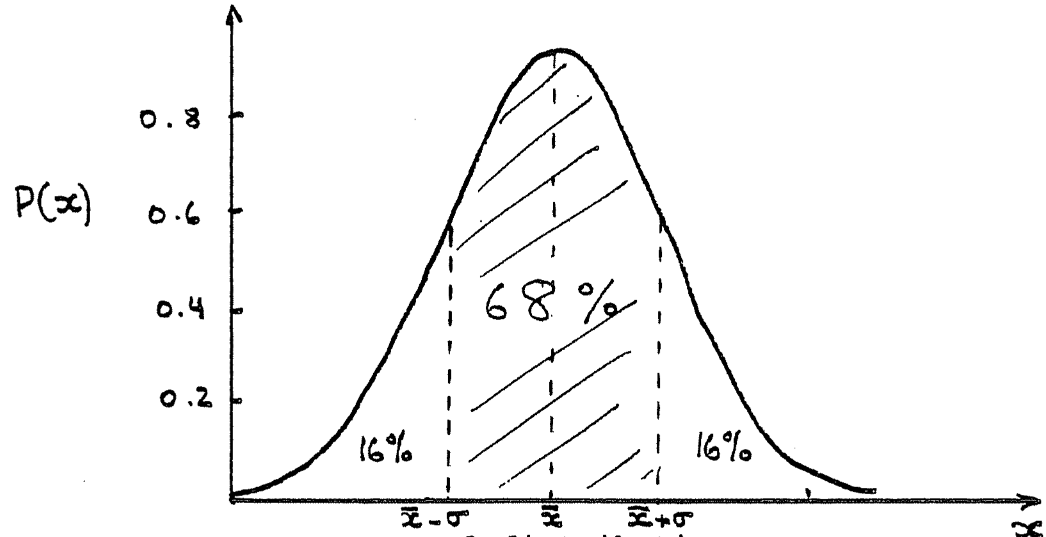


Figure 5. Normal distribution

b) Evaluation of σ for Various Distributions

i) Bionomial distribution

$$\sigma^2 = \sum_{x=o}^{x=z} (x-zp)^2 P(x) = \sum_{x=o}^{x=z} \frac{(x-zp)^2 \; z! \; p^x \; (1-p)^{z-x}}{x! \; (z-x)!} \tag{9}$$

$$\rightarrow zp\,(1-p) = m\,(1-p)$$

ii) Normal distribution

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-m)^2 \,(P(x)\; dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-m)^2 \; e^{-(x-m)^2/2\sigma^2} \, dx \equiv \sigma^2 \tag{10}$$

iii) Poisson distribution

$$\sigma^2 = \sum_{x=o}^{x=\infty} \frac{(x-m)^2 m^x}{x!} \; e^{-m} = m \tag{11}$$

Note that in this case the standard deviation of the distribution of individual observations is simply the square root of the mean, $\sqrt{m}$. This agrees with the result for the bionomial distribution in limit $p << 1$.

c) <u>Estimate of standard deviation from a finite series of observations</u>

In a finite set of observations, we can never know m exactly, hence we cannot determine σ exactly. The best approximation for the standard deviation of the distribution in terms of the finite number n of observations is

$$\sigma^2 \equiv \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right] = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \qquad (12)$$

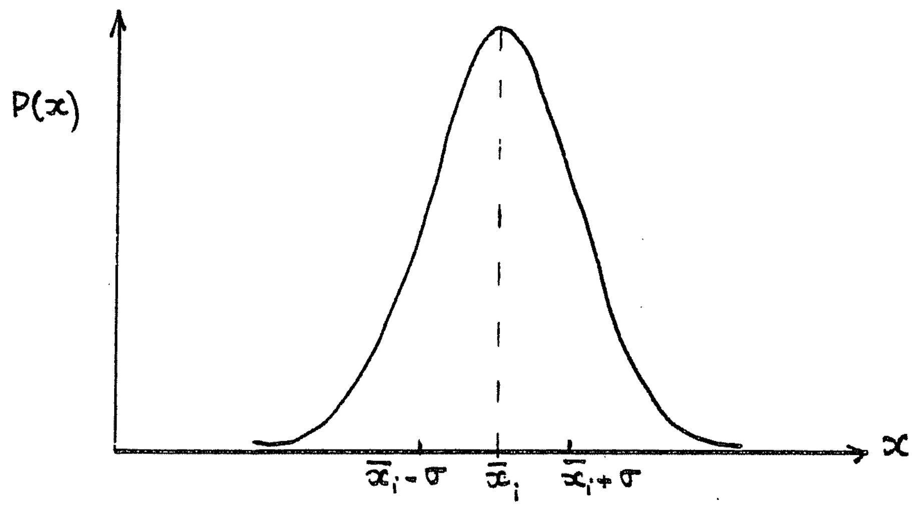
The reason the factor n-1 comes into this result is that out of n measurements (or n degrees of freedom) we have used one to estimate the mean $\bar{x}$ and therefore we only have n-1 datum points left to determine the standard deviation. If we only made one measurement $x = \bar{x}$, and σ would be indeterminate.

d) <u>Standard deviation of the mean value:</u> $\sigma_{\bar{x}}$ <u>(Standard error)</u>

If the n individual measurements of x exhibit an approximate normal distribution about the mean value $\bar{x}$, then 68% of the individual observations have fallen within the central band $\bar{x} \pm \sigma$. <u>One</u> additional <u>single</u> observation would have a 68% chance of lying within $\bar{x} \pm \sigma$. Thus, σ the standard deviation of the distribution is called more precisely "the standard deviation of a <u>single</u> observation."

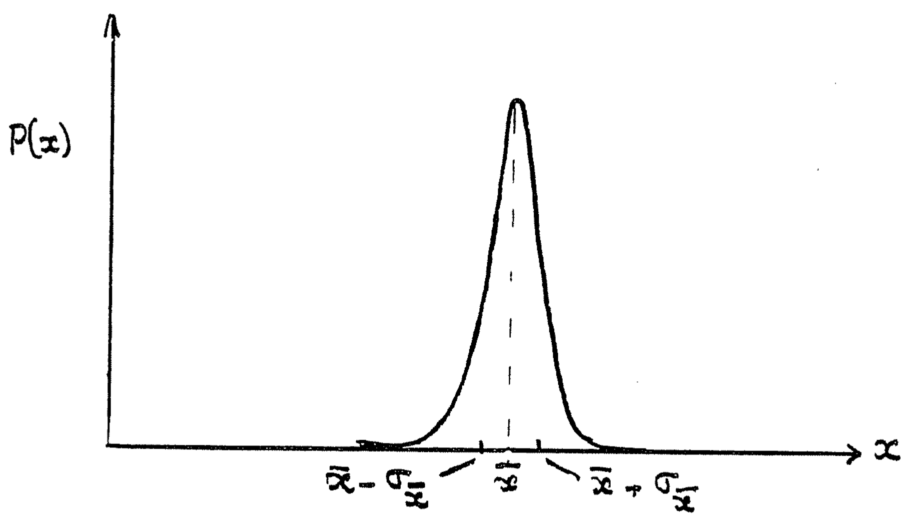
If we repeat the entire experiment of n observations, we would expect the new mean value to have a much greater than 68% chance of falling within $\bar{x} \pm \sigma$. Therefore, in reporting the mean value $\bar{x}$, we will assign it a standard deviation $\sigma_{\bar{x}}$ which implies that there is approximately a 68% chance than some new mean value $\bar{x}_2$ will live within the band $(\bar{x} \pm \sigma_{\bar{x}})$. Obviously $\sigma_{\bar{x}} < \sigma$.

Consider a series of k means $\bar{x}_1 \ldots \bar{x}_k$ each based on n observations. They will exhibit a normal distribution about the grand average $\bar{x}$.



Distribution of n single measurements.

Figure 6.



Distribution of k measurements of the mean $\bar{x}$.

Figure 6.

Then, assuming $\frac{1}{n}\sum_{i=1}^{n} x_{ij} \approx m$ for all j and $\frac{1}{n}\sum_{i=1}^{n} (x_{ij}-m)^2 \sim \sigma^2$ for all j

$$\sigma_{\bar{x}}^2 = \frac{1}{k}\sum_{j=1}^{j=k} (\bar{x}_j - m)^2 = \frac{1}{k}\sum_{j=1}^{k}\left[\frac{\Sigma x_{ij}}{n} - m\right]^2 = \frac{1}{k}\sum_{j=1}^{k}\frac{1}{n^2}\left[\sum_{i=1}^{n}(x_{ij}-m)\right]^2 \approx \frac{\sigma^2}{n} .$$

Thus the result of a single series of n measurements of x is to be reported as $\bar{x} \pm \sigma_{\bar{x}}$, where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{14}$$

$$\sigma_{\bar{x}} \sim \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{1}{n(n-1)}\sum_{1}^{n}(x_i - \bar{x})^2} \tag{15}$$

A repetition of the series of n measurements would in general give a different mean value, but the chance that the new mean value would lie within $\bar{x} \pm \sigma_{\bar{x}}$ is 68%.

e) <u>Probable error</u>

The standard deviation has a definite statistical meaning. However, one might want to know what is the probability that a particular measurement has an equal probability of being outside or inside a limit set by the "probable error" r.

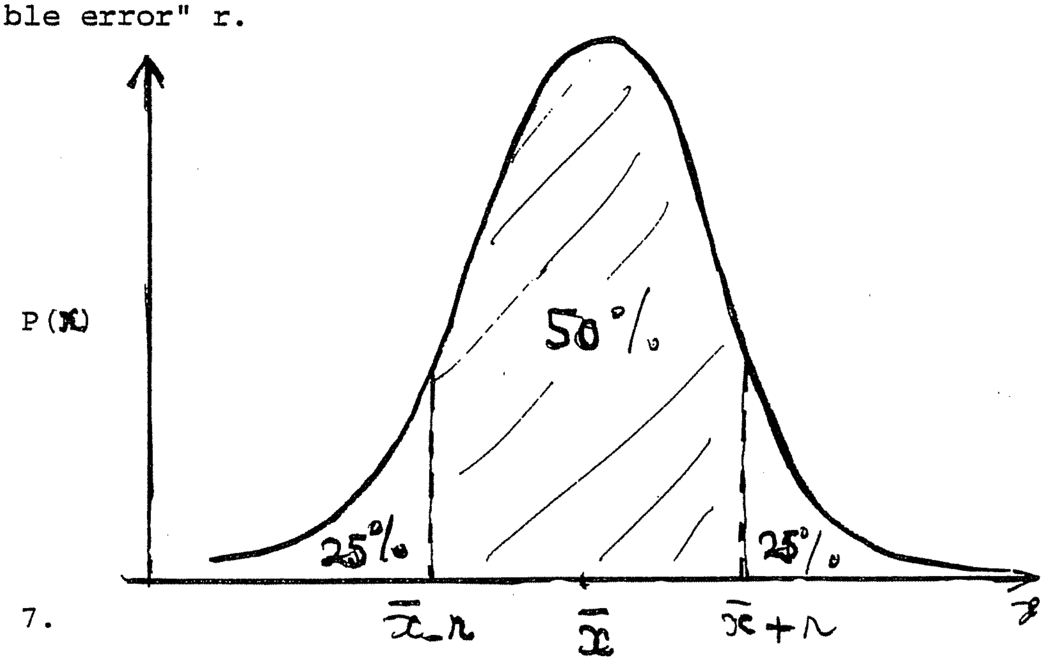


Figure 7.

Thus, for a normal distribution, the probable error is by definition exactly as likely to be exceeded as not, or in other words, the chance that $|m-\bar{x}|$ is greater than $r_{\bar{x}}$ is 0.5.

$$r = 0.6745\ \sigma$$
$$r_{\bar{x}} = 0.6745\ \sigma_{\bar{x}}$$

C) "Weighted" Measurements

Some observations are more reliable than others and therefore should influence the result more than the others. A weight w_i can be assigned to each measurement of x_i. Under such conditions, weighted means and standard deviations must be evaluated

$$\bar{x} = \sum_{i=1}^{n} \frac{w_i\ x_i}{\sum_{i=1}^{n} w_i} \tag{16}$$

and $$\sigma_{\bar{x}} = \sqrt{\frac{\Sigma\ w_i\ (x_i - \bar{x})^2}{(n-1)\ \Sigma\ w_i}} \tag{17}$$

If $w_i=1$, these expressions reduce to the ones obtained above.

In estimating the standard deviation of the mean of a set of observations, it is useful to note that the appropriate weights are related to the standard deviations of individual measurements, namely $w_i \equiv \frac{1}{\sigma_i{}^2}$

$$\text{Thus} \qquad \bar{x} = \frac{\Sigma(x_i / \ _i{}^2)}{\Sigma(1/\sigma_i^2)} \tag{18}$$

$$\text{and} \qquad \sigma_{\bar{x}} = \sqrt{\frac{1}{\Sigma \ 1/\sigma_i^2}} \tag{19}$$

PROPAGATION OF ERRORS

In general, if the quantity $x = f\ (u, v\ldots)$ to be determined is a function of various observables $(u, v\ldots)$ each of which is determined with some uncertainty $(\sigma_u, \sigma_v\ldots)$, the error in the quantity can be evaluated from the following assumptions (see Bevington, page 58, for more precise details).

$$x = f\ (u, v, w\ldots) \text{ and assuming}$$
$$\bar{x} = f(\bar{u}, \bar{v}, \bar{w}\ldots)$$

$$\text{Then} \quad dx = \frac{\partial f}{\partial u}\, du + \frac{\partial f}{\partial v}\, dv \ \ldots \tag{20}$$

$$\text{and} \quad (dx)^2 = \left(\frac{\partial f}{\partial u}\, du\right)^2 + 2\, \frac{\partial f}{\partial u}\, \frac{\partial f}{\partial v}\, du\, dv + \left(\frac{\partial f}{\partial v}\, dv\right)^2 + \ldots \tag{21}$$

If the measurements of u, v, w... are independent, the cross terms will average to zero and the uncertainty in x becomes

$$(dx)^2 = \left(\frac{\partial f}{\partial u}\, du\right)^2 + \left(\frac{\partial f}{\partial v}\, dv\right)^2 + \ldots \tag{22}$$

$x = au$	$\sigma = a\sigma_u$
$x = u + v + \ldots$	$\sigma = \sqrt{\sigma_u^{\ 2} + \sigma_v^{\ 2} + \ldots}$
$x = uvw$	$\frac{\sigma}{\bar{x}} = \sqrt{(\frac{\sigma_u}{\bar{u}})^2 + (\frac{\sigma_v}{\bar{v}})^2 + (\frac{\sigma_w}{\bar{w}})^2 + \ldots}$
$x = \frac{u}{v}$	$\frac{\sigma}{\bar{x}} = \sqrt{(\frac{\sigma_u}{\bar{u}})^2 + (\frac{\sigma_v}{\bar{v}})^2}$
$x = u^n$	$\frac{\sigma}{\bar{x}} = n\ \frac{\sigma_u}{u}$
$x = \ln u$	$\sigma = \frac{\sigma_u}{u}$

EXAMPLES:

1) <u>Standard deviation of an average</u>

Four observations of the ratio $x = e/m$ of the charge to mass of an electron were made. Compute the mean and the standard deviation.

Observation	$x \equiv e/m$ $(\cdot 10^{-7})$	$x - \bar{x}$ $(\cdot 10^{-7})$	$(x - \bar{x})^2$ $(\cdot 10^{-14} \cdot 10^{-8})$
	1.7610	− 0.0019	361
	1.7587	+ 0.0004	16
	1.7571	+ 0.0020	400
	1.7597	− 0.0006	36
	$\bar{x} = 1.7591 \cdot 10^{-7}$		$\Sigma = 813 \cdot 10^{-22}$

$$\bar{x} = \frac{\sum_{i=1}^{4} x_i}{4} = 1.7591 \cdot 10^{-7}$$

$$\sum_{i=1}^{4} (x_i - \bar{x})^2 = 813 \cdot 10^{-22}$$

$$\sigma_x = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n-1}} = \text{standard deviation in any one of the measurements } x_i$$

$$\sigma_{\bar{x}} = \sqrt{\frac{\sum_i (x_i - \bar{x})^2}{n(n-1)}} = \text{standard deviation in the mean } \bar{x}$$

$$= \sqrt{\frac{1}{4 \cdot 3} (813 \cdot 10^{-22})} =$$

$$= 0.000823 \cdot 10^{-7}$$

Result:

$$\frac{e}{m} = (1.7591 \pm 0.0008) \times 10^{-7} \text{ emu}$$

2) <u>Standard deviation of a function</u>

Consider the measurement of the electron charge e from a series of observations of falling charged oil drops. In this experiment the electron charge is given by

$$ne = \underbrace{400\,\pi s^{3/2}\; d\; \frac{(9\,\eta/2)^{3/2}}{g(\mu-\rho)^{1/2}}}\;\underbrace{\left(\frac{1}{1+\frac{b}{pa_1}}\right)^{3/2}}\;\underbrace{\left(\frac{\frac{1}{t_d}+\frac{1}{t_u}}{V\sqrt{t_d}}\right)} \qquad (23)$$

$$= A \cdot B \cdot nC$$

Where: s = distance (actual) of rise and fall of drops in cm.
d = plate separation of condenser, in cm.
g = acceleration due to gravity = (980.3 0.1) cm sec^{-2}
η = viscosity of air in c.g.s. units
ρ = density of air in g cm^{-3}
μ = density of oil in g cm^{-3}
b = constant =0.000617
p = atmospheric pressure in cm Hg

} constant throughout exp.

drop.
a_1 = approximate radius of drop $=\left(\frac{9\eta\; s}{2g(\sigma-\rho)\; t_d}\right)^{1/2}$

} constant for one drop.

t_d = time of fall under gravity, sec.
t_u = time of rise with elec. field, sec.
V = voltage on condenser, volts.
n = integer = number of electron charges on drop.

} constant for a given charge on a drop.

The data are obtained as follows: with a given drop and a fixed charge, a set of several observations of t_d, t_u, and V is obtained. A series of such sets of observations of t_d, t_u, and V set corresponding to a different charge on the same drop. Finally several such series of data are collected, each series having to do with a different drop.

From each set of observations average values and standard deviations can be calculated from Eqs. 6 and 15. The charge e can then be found from Eq. (23), and its standard deviation can be calculated from Eq. (22).

The following analysis can be used conveniently. The first bracket term A, is a constant throughout the experiment. A and its standard deviation can be evaluated first:

$$A = 400\,\pi \cdot \frac{d\, s^{3/2} (9\eta/2)^{3/2}}{g\,(\mu - \rho)^{\frac{1}{2}}} \qquad (24)$$

Since ρ amounts to only 0.1% of μ, any uncertainty in ρ will affect A by a completely negligible amount and will be neglected. Then

$$\frac{\sigma_A}{A} = \sqrt{\left(\frac{3}{2}\,\frac{\sigma_s}{s}\right)^2 + \left(\frac{3}{2}\,\frac{\sigma_\eta}{\eta}\right)^2 + \left(\frac{1}{2}\,\frac{\sigma_g}{g}\right)^2 + \left(\frac{1}{2}\,\frac{\sigma_\mu}{\mu}\right)^2 + \left(\frac{\sigma_d}{d}\right)^2} \qquad (25)$$

The term $B = \left(\dfrac{1}{1+\dfrac{b}{pa_1}}\right)^{3/2}$ differs from unity by only a few percent. Thus, the uncertainty in B can be neglected. It is fortunate that $B \approx 1$, because since a_1 is a function of t_d, it would in fact be quite complicated to calculate the standard deviation in $B \cdot nC$.

The remaining term in the expression for ne is nC

$$nC = \frac{1}{V}\left\{t_d^{-3/2} + t_u^{-1}\ t_d^{-1/2}\right\} \qquad (26)$$

An nC should be calculated for each set of observations using average values of t_d, t_u and V and the appropriate integer n.

$$\sigma_{nC} = \sqrt{\left[\left(\frac{\partial nC}{\partial V}\sigma_V\right)\right]^2 + \left(\frac{\partial nC}{\partial t_d}\quad \sigma_{t_d}\right)^2 + \left(\frac{\sigma nC}{\sigma t_u}\quad \sigma t_u\right)^2} \qquad (27)$$

$$= \sqrt{\left(\frac{nC}{V}\ \sigma_V\right)^2 + \left[\frac{1}{2V(t_d)^{3/2}}\left(\frac{3}{t_d} + \frac{1}{t_u}\right)\sigma_{t_d}\right]^2 + \left[\frac{1}{Vt_d^{\frac{1}{2}}\ t_u^2}\quad \sigma_{t_u}\right]^2}$$

However, two other uncertainties have been omitted up to this point, namely the uncertainty in the calibration of the stopwatch, σ_w and the uncertainty in the calibration of voltmeter σ_m. Including these we obtain the standard deviation

$$\frac{\sigma_{ne}}{ne} = \sqrt{\left(\frac{\sigma_A}{A}\right)^2 + \left(\frac{\sigma_{nC}}{nC}\right)^2 + \left(\frac{3}{2}\ \frac{\sigma_w}{\bar{t}}\right)^2 + \left(\frac{\sigma_m}{\bar{V}}\right)^2} \qquad (28)$$

Here $\bar{t}$ is about the average time recorded and $\bar{V}$ about the average voltage. This is obviously an approximation but it gives more reliable values of $\sigma_{\bar{e}}$ than if these calibration uncertainties had been included in each of the nC. The 3/2 factor comes from the fact that in both parts of nC, "time" occurs to the (-3/2) power).

Finally,

$$ne = A \cdot B \cdot nC \pm \sigma_{ne}$$

All of the foregoing procedure for handling the data for "e" is shown schematically in the following table:

	Series for Drop A			Series for Drop B		
	Set 1	Set 2	Set 3	Set 1	Set 2	Set 3
	charge n_1	charge n_2	etc.	charge n_{10}	charge n_{11}	etc.
	- - - -	- - - -	etc.	- - - -	- - - -	etc.
$\bar{t}_u$ =	( ) ± ( )	( ) ± ( )	etc.	( ) ± ( )	( ) ± ( )	etc.
	- - - - -	- - - - -		- - - - -	- - - - -	
$\bar{t}_d$ =	( ) ± ( )	( ) ± ( )	etc.	( ) ± ( )	( ) ± ( )	etc.
	- - - -	- - - -		- - - -	- - - -	
V =	( ) ± ( )	( ) ± ( )	etc.	( ) ± ( )	( ) ± ( )	etc.

Calculate nC $n_1 C_1 \pm (\ )$, $n_2 C_2 \pm (\ )$ etc. .... $n_{10} C_{10} \pm (\ )$, $n_{11} C_{11} \pm (\ )$

n_1 $\quad$ n_2 $\quad$ etc. .... $\quad$ n_{10} $\quad$ n_{11} $\quad$ Find the approx. value of C, the greatest common divisor of the (nC)'s. Then find the value of the integers n_1, n_2, etc.

$C_1 \pm (\ )$ $\quad$ $C_2 \pm (\ )$ $\quad$ etc. .... $\quad$ $C_{10} \pm (\ )$ $\quad$ $C_{11} \pm (\ )$ $\quad$ Divide and find C_1, C_2, etc.

$\bar{C} \pm \sigma_{\bar{C}}$ $\qquad$ The observational uncertainties of the experiment.

$\downarrow$

$$\boxed{\begin{array}{l}\text{Combine with } A \pm \sigma_A \text{ and} \\ \text{Calibration uncertainties in } \bar{t} \text{ and V.}\end{array}}$$

$\downarrow$

$\bar{e} \pm \sigma_{\bar{e}}$

3) Nuclear counting statistics

Consider the number of particles emitted from a constant source of radiation in an counting interval Δt. As many measurements are taken the individual values will be different, but will be distributed about a mean according to a Poisson distribution.

If $\bar{n}$ is the average number of particles arriving in a certain time, and n is the actual number arriving in this time, the probability of finding this number is given by

$$P_n = \frac{(\bar{n})^n e^{-\bar{n}}}{n!} \qquad (29)$$

The standard deviation of a single count of n particles is given by

$$\sigma^2 = \sum_{o}^{\infty} (n-\bar{n})^2 \; P_n = \bar{n} \qquad (30)$$

But for a single measurement $n=\bar{n}$, therefore for a single measurement $\sigma = \sqrt{n}$, i.e., the uncertainty is equal to the square root of the number of counts.

Consider, for example, the experimental measurements of the activity of a radioactive source. The data tabulated below are the number of counts x_i detected in each time interval Δt_i

Interval t_i, min	Counts x_i
1	14
1	28
1	27
1	16
1	13
1	14
1	30
1	15
1	27
1	17
10	220
20	421

$$\bar{x} = \frac{1}{N} \Sigma x_i = 201 \text{ counts/10 min} = 20.1 \text{ counts/min}$$

$$\sigma \simeq \sqrt{\bar{x}} = 4.5 \text{ counts/1 min} = 4.5 \text{ counts/min}$$

$$\sigma_{\bar{x}} \equiv \simeq \sqrt{\frac{\bar{x}}{N}} = \quad 2.01 = 1.4 \text{ counts/min}$$

The activity of the radioactive source is measured $N = 10$ times with a time interval $\Delta t \simeq 1$ min. The average of these data points is $\bar{x} \simeq 20.1$ counts/min. The spread of the data points is characterized by $\sigma \simeq 4.5$ counts/min calculated according to Equation 39. The uncertainty in the mean is calculated according to $\sigma_{\bar{x}} \simeq \frac{\sigma}{\sqrt{N}} = 1.4$ counts/min.

If we were to combine the data into one observation $x' = \Sigma x_i$ for one 10-min interval, we would obtain the same result. The activity is $x' = 201$ counts/10 min = 20.1 counts/min as before. The uncertainty in the result is given by the standard deviation of the single data point $\sigma_{x'} = \sigma' \simeq 14$ counts/10 min = 1.4 counts/min.

If we were to make an additional measurement $x_{11} \simeq 220$ counts/min for a 10-min period, we could combine x' and x_{11} exactly as above to obtain a total $x_r = x' + x_{11} = 21$ counts/min with an uncertainty $\sigma_{\mu T} = \sqrt{x_T} - 20.5$ counts/20 min = 1.0 count /min which is smaller than $\sigma_{\bar{x}}$ by a factor of $\sqrt{2}$. We could, alternatively, combine the original data points x_i with the new data point x_{11} to evaluate the average $\bar{x}_T$:

$$x_T = 201 + 220 = 421 \text{ counts/20 min} = 21.0 \text{ counts/min}$$

Or $$\bar{x}_T = \frac{\Sigma(x_i/\sigma_i^2)}{\Sigma(1/\sigma_i^2)} = \frac{201/20.1 + 22/2.2}{10/20.1 + 1/2.2} \text{ counts/min}$$
$$= 21.0 \text{ counts/min}$$

The uncertainty in the final result σ_r comes from combining the uncertainties of the individual data points

$$\sigma_{11} \simeq \sqrt{220} = 15 \text{ counts/10 min} = 1.5 \text{ counts/min}$$

And $$\sigma_T = \left(\frac{10}{\sigma^2} + \frac{1}{\sigma_{11}^2}\right)^{\frac{1}{2}} \simeq \left(\frac{10}{20.1} + \frac{1}{2.2}\right)^{-\frac{1}{2}} = 1.0 \text{ count/min}$$

Note that we could have simplified matters greatly by considering all the data as one experimental point: $x = 421$ counts/20 min = 21 counts/min with an uncertainty given by the square root of the total number of counts $\sigma_x \simeq \sqrt{421} = 20.5$ counts/20 min = 1.0 count/min.

METHOD OF LEAST SQUARES FIT

We often wish to determine the characteristic of an experimentally measured quantity, say y, as a function of some other quantity x. For example, we want to determine the temperature T along a metal rod suspended between two constant temperature baths as a function of the position along the rod. The data are indicated by the figure below and represent N measurements of the temperature at N different positions x_i along the rod.

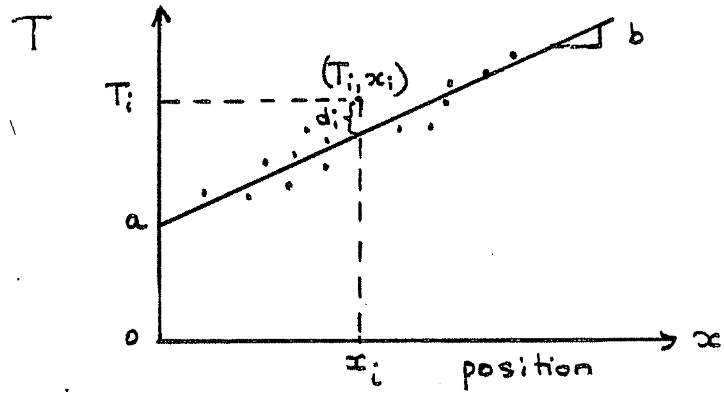


Figure 8.

Let us assume that T = a + bx and let us find the values of the parameter a and b that best fit the data. The method of least square is used for this purpose. The deviation of the procedure is well described in Bevington and Evans. Only the procedural steps are given here:

First calculate χ^2

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{\Delta y_i}{\sigma_i} \right)^2 = \sum_{i=1}^{N} \frac{1}{\sigma_i^2} (y_i - a - bx_i)^2$$

χ^2 is a measure of the deviation of the data from the proposed formula. Then one wants to minimize χ^2 with respect to the parameter a and b by setting.

$$\frac{\partial \chi^2}{\partial a} = 0 \quad \text{and} \quad \frac{\partial \chi^2}{\partial b} = 0$$

Assume for simplicity that all $\sigma_i = \sigma$ are equal. Then

$$\frac{\partial \chi^2}{\partial a} = -\frac{2}{\sigma^2} \Sigma \ (y_i - a - bx_i) = 0$$

$$\frac{\partial \chi^2}{\partial b} = -\frac{2}{\sigma^2} \Sigma \ x_i \ (y_i - a - bx_i) = 0$$

Solve these equations for a and b

$$a = \frac{\Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i}{\Delta}$$

$$b = \frac{N \Sigma x_i y_i - \Sigma x_i y_i}{\Delta}$$

and $\Delta = N \Sigma x_i^2 - (\Sigma x_i)^2$

When $\sigma \neq \sigma_i$

$$a = \frac{1}{\Delta}\left(\Sigma \frac{x_i^2}{\sigma_i^2} \Sigma \frac{y_i}{\sigma_i^2} - \Sigma \frac{x_i}{\sigma_i^2} \Sigma \frac{x_i y_i}{\sigma_i^2}\right)$$

$$b = \frac{1}{\Delta}\left(\Sigma \frac{1}{\sigma_i^2} \Sigma \frac{x_i y_i}{\sigma_i^2} - \Sigma \frac{x_i}{\sigma_i^2} \Sigma \frac{y_i}{\sigma_i^2}\right)$$

and $$\Delta = \Sigma \frac{1}{\sigma_i^2} \Sigma \frac{x_i^2}{\sigma_i^2} - \Sigma \left(\frac{x_i}{\sigma_i}\right)^2$$

The errors in the parameters can be estimated from the following

$$\sigma_z = \Sigma \sigma_i^2 \left(\frac{\partial z}{\partial y_i}\right)^2 \qquad \text{where } z = a, b$$

Again, when $\sigma_i = \sigma$

$$\frac{\partial a}{\partial y_j} = \frac{\Sigma x_i^2 - x_j \Sigma x_i}{\Delta}$$

and

$$\frac{\partial b}{\partial y_j} = \frac{N x_j - \Sigma x_i}{\Delta}$$

$$\Delta = N\Sigma x_i^2 - (\Sigma x_i)^2$$

Thus

$$\sigma_a^2 \sim \sum_{j=1}^{n} \frac{\sigma^2}{\Delta^2} \left[(\Sigma x_i^2)^2 - 2x_j \Sigma x_i^2 \Sigma x_i + x_j^2 (\Sigma x_i)^2\right]$$

$$= \frac{\sigma^2}{\Delta^2} \left[N(\Sigma x_i^2)^2 - 2(\Sigma x_i)^2 \Sigma x_i^2 + \Sigma x_i^2 (\Sigma x_i)^2\right]$$

$$= \frac{\sigma^2}{\Delta^2} (\Sigma x_i^2) \left[N\Sigma x_i^2 - (\Sigma x_i)^2\right] = \frac{\sigma^2}{\Delta} \sum x_i^2$$

$$\sigma_b^2 \sim \sum_{j=1}^{M} \frac{\sigma^2}{\Delta^2} \left[N^2 x_j^2 - 2 x_j \Sigma x_i + (\Sigma x_i)^2\right]$$

$$= \frac{\sigma^2}{\Delta^2} \left[N^2 \Sigma x_i^2 - 2N(\Sigma x_i)^2 + N(\Sigma x_i)^2\right]$$

$$= \frac{N\sigma^2}{\Delta^2} \left[N \Sigma x_i^2 - (\Sigma x_i)^2\right] = \frac{N\sigma^2}{\Delta}$$

In cases when there are "experimental" deviations $\sigma_i(x_i)$ in the independent quantity x, in addition to the "statistical" deviation $\sigma_i(y_i)$ in the dependent quantity y_i, the root mean squares of the standard deviations for the experimental deviations $\sigma_i(x_i)$ and the statistical deviations $\sigma_i(y_i)$ must be used in the calculations:

$$\sigma_i = \sqrt{\sigma_i^2(x_i) + \sigma_i^2(y_i)} \quad .$$