Lecture 4

Minkowski space-time diagram (4 dimension)

\[ (ct = -x, ct = x) \]

Future \[ \uparrow \]

Present \[ \downarrow \]

Past \[ \downarrow \]

World line \& path between 2 events in space-time.

Invariant space-time distance: \[ S^2 = x^2 + y^2 + z^2 - ct^2 \]
e.g. \[ S^2 = x^2 - (ct)^2 \] Lorentz Trans. \[ S'^2 = S^2 \]

\[ S^2 = x^2 - (ct)^2 \]

\[ S^2 = x^2 - (ct')^2 \]

Space-time interval: \[ AS^2 = dx^2 + dy^2 + dz^2 - c^2 (dt)^2 \] is also invariant.

\[ AS^2 > 0 \] : light-like interval. e.g. EF
\[ AS^2 < 0 \] : i.e. \( x^2 > ct^2 \) space-like interval (e.g. CD).
\[ AS^2 < 0 \] : i.e. \( x^2 < ct^2 \) time-like interval. e.g. AB.

Since \( AS^2 \) is Lorentz invariant, time-like world line (necessary for casual relationship of 2 events) is Lorentz invariant.

\[ \Rightarrow \] Casualty is to reality invariant!

Twin Paradox.

The reciprocal effect of time dilation between inertial frames.

Frank \[ \Rightarrow \] Many

\[ E \]

Frank: Many is in moving frame, so she is younger.
Mary: Frank is in moving frame, so he is younger.
Lecture 4.

the key difference: Many \((r/space shuttle)\) experience acceleration at the turning point. (remote star). So other frame is NOT one inertial frame through out the whole trip.

For Mary: length contraction \(L' = \frac{1}{\gamma}\)

First half: Mary is on frame \(k' + v\)

2nd Half: “on \(k' - v\)

Travel time. Time for Mary: \(t' = \frac{2L}{v} \gamma = \frac{2L}{v}\gamma \left(1 - \beta^2\right)^{-1/2}\)

At turning point. \(x = L, \quad \gamma \frac{t}{v} = \frac{1}{2}L\left(1 - \beta^2\right)^{-1/2}\)

Simultaneity before: \(ct = px + b_1 \Rightarrow b_1 = \frac{c}{v} - \beta L = \frac{c}{v} \left(1 - \beta^2\right)^{1/2}\)

After: \(ct = px + b_2 \Rightarrow b_2 = \frac{c}{v} \beta L = \frac{c}{v} \left(1 - \beta^2\right)^{1/2}\)

the difference: \(\Delta b = b_2 - b_1 = \frac{2L}{v} \beta^2\)

For many, Frank's clock "suddenly" jumps from \(\frac{t}{\gamma^2 v^2}\) to \(\frac{1}{v} \left(1 + \beta^2\right)\) at the turning point.

except the turning point, Mary's perception of Frank's clock is \(\gamma\) times slower, so i.e. \(\frac{t}{\gamma^2 v^2} = \frac{2L}{v^2}\gamma\)

the total travel time of the round trip for Frank's clock is:
\[
\gamma \frac{t}{v} = \gamma \frac{2L}{v} \left(1 - \beta^2\right)^{-1/2} = \frac{2L}{v} \gamma \left(1 - \beta^2 + \beta^4\right)^{1/2}
\]

No paradox.

Further analysis of twin paradox, see Table 2.1 on Page 48.

2.11 Relativistic Momentum.

Newton's Law, momentum conservation, energy conservation should be invariant under Lorentz transformation. Thus momentum & energy needs to redefine in Special Relativity.

E.g. momentum conservation:

\[
p = mv = 2mU' \Rightarrow U' = \frac{v}{2} \frac{v}{\gamma}
\]
However in the $k'$ frame, \( v_1 = \frac{v - v'}{1 - \frac{v v'}{c^2}} \), \( v_2 = -v' \pm \frac{v}{2} \)

\[
\implies p_1 = m v_1 + m v_2 = m v \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \neq 0 \quad p_2 = 0.
\]

Momentum is not conserved in $k'$ frame.

* Relativistic momentum: \( \vec{p} = \gamma m \vec{u} \) here \( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \).

* Newton's law \( \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} (\gamma m \vec{u}) \). \( \gamma m \): relativistic mass.

\( \gamma m \vec{u} \) for faster object, it is harder to accelerate.

* Relativistic Energy.

Let's compute the kinetic energy gain of an object at rest under a constant force (work-energy theorem).

\[
k.E. = W = \int_0^u F \, dx = \int_0^u \frac{d}{dt} (\gamma m \vec{u}) \, dx \quad \text{Note } dx = u \, dt.
\]

\[
= m \int_0^u \left. \frac{d}{dt} (\gamma m \vec{u}) \right| \bigg|_0^u \, dt - \int_0^u (\gamma m \vec{u} \cdot du)
\]

\[
= m \left[ \gamma u^2 + c^2 \sqrt{1 - \left( \frac{u}{c} \right)^2} \bigg|_0^u \right]
\]

\[
= m \left[ \gamma u^2 + c^2 \sqrt{1 - \left( \frac{v}{c} \right)^2} \right]
\]

\[
= m \left[ \gamma (c^2 - v^2) \right]
\]

\[
= m c^2 (\gamma - 1).
\]

* Low speed limit, \( \beta \ll 1 \). \( \text{KE} \approx \frac{1}{2} m \beta^2 \Rightarrow \text{KE} \approx \frac{1}{2} mc^2 \cdot \beta^2 = \frac{1}{2} m u^2 \).

Reduced to the expression in classical mechanics.

* If we consider KE = Total energy - Rest energy, then, \( T.E. = \gamma mc^2 \) rest Energy = \( mc^2 \). \( m \): rest mass.

\( c^2 \) is a huge \( \approx 9 \times 10^{16} \text{ m}^2/\text{s}^2 \).

E.g. 1kg mass \( \approx 9 \times 10^{16} \text{ J} \approx 2.5 \times 10^9 \) (kwh) enough to power the whole world for 2 hours!!

* Any form of Energy (Potential, chemical, nuclear, etc.) \( \Rightarrow \) works
  * Mass-energy conversion is not trivial.
    * e.g. nuclear binding energy (fission & fusion)
      * Anti-matter & matter annihilation \( (e^- + e^+ \rightarrow 2 \gamma) \)

- Relationship of relativistic energy & momentum:
  \[ E = \gamma mc^2 \quad \text{and} \quad \mathbf{p} = \gamma \mathbf{m} \mathbf{u} \] (imply massive particles, i.e. \( m \neq 0 \)).

What's the dispersion relation of relativistic particles? \( \mathbf{E}(\mathbf{p}) = ? \)

Since \( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \gamma^2 = \frac{1}{1 - \beta^2} \Rightarrow \gamma^2 = 1 + \frac{v^2}{c^2} \)

\[ \gamma (mc^2)^2 = (mc^2)^2 \left( 1 + \frac{v^2}{c^2} \right)^2 = (mc^2)^2 \left( 1 + \frac{v^2}{c^2} \right) \]

\[ \Rightarrow E^2 = m^2 c^4 + p^2 c^2 \quad (\mathbf{p} \cdot \mathbf{p} = p^2 = \gamma^2 m^2 c^2 \quad \text{substitute} \quad \gamma E) \]

\[ E = \pm \sqrt{p^2 c^2 + m^2 c^4} \]

[Here, "+" is for particle (\( \Psi \)), "-" is for anti-particle (\( \bar{\Psi} \)).

E.g. electron and its anti-particle, positron.

- Low energy limit (\( \beta \ll 1 \)): \( p \ll mc \)

\[ E \approx \sqrt{m^2 c^4 \left( 1 + \left( \frac{p^2}{m^2 c^2} \right) \right)} \approx mc^2 \left( 1 + \frac{p^2}{2m^2 c^2} \right) = mc^2 + \frac{p^2}{2m} \]

Sometimes it's called the classical limit. Rest mass \( m = \text{classical mechanics} \)

* In classical mechanics, \( mc^2 \) is often neglected (an offset) because it's just a constant.

- High energy limit (\( \beta \gg 1 \)): \( \text{i.e. } p \gg mc \)

\[ E \approx pc \]

Sometimes, it's called the relativistic limit.

[Note: for particles of light, photons \( E = pc \) for any \( p \), truly massless.]