34. Normalization 

\[ 1 = \int_{-\infty}^{\infty} \psi^* \psi \, dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} \, dx = 2A^2 \int_{0}^{\infty} x^2 e^{-\alpha x^2} \, dx = 2A^2 \frac{1}{4\alpha} \sqrt{\frac{\pi}{\alpha}}. \]

Solving for \( A \) we find \( A = \sqrt{2\pi^{1/4}} \alpha^{3/4} \). \( \langle x \rangle = A^2 \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2} \, dx = 0 \) because the integrand is odd over symmetric limits. \( \langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} \, dx = \frac{3}{4} A^2 \pi^{1/2} \alpha^{-3/2} = \frac{3}{2\alpha} \); \n
\[ \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3}{2\alpha}}. \]

36. Taking the second derivative of \( \psi \) for the Schrödinger equation:

\[ \frac{d^2 \psi}{dx^2} = 5A\alpha x e^{-\alpha x^2/2} - 2\alpha^2 A Ax^3 e^{-\alpha x^2/2}; \]

\[ \frac{d^2 \psi}{dx^2} = (5\alpha - 5\alpha^2 x^2 - 6\alpha x^3 + 2\alpha^3 x^4) A e^{-\alpha x^2/2}. \]

Starting with Equation (6.56), and with the wave function given in the problem, we have:

\[ \frac{d^2 \psi}{dx^2} = (\alpha^2 x^2 - \beta) \psi - (\alpha^2 x^2 - \beta) A(2\alpha x^2 - 1)e^{-\alpha x^2/2} \]

\[ = (2\alpha^3 x^4 + x^2 (-2\alpha \beta - \alpha^2)) + \beta) A e^{-\alpha x^2/2}. \]

Matching our two values of \( \frac{d^2 \psi}{dx^2} \) we see that the Schrödinger equation can only be satisfied if \( \beta = 5\alpha \). Then \( \frac{2mE}{\hbar^2} = 5 \sqrt{\frac{mk}{\hbar^2}} \) or \( E = \frac{5}{2} \hbar \omega \). This is the expected result, because the wave function contains a second-order polynomial (in \( x \)), and with \( n = 2 \) we expect \( E = \left( n + \frac{1}{2} \right) \hbar \omega = \frac{5}{2} \hbar \omega \).
39. The classical frequency for a two-particle oscillator is [see Chapter 10, Equation (10.4)]

\[ \omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k}{m_1 + m_2}} \frac{1}{m_1 m_2} \sqrt{2k/m} \]

since the masses are equal in this case. The energies of the ground state \((E_0)\) and the first three excited states are given by

\[ E_n = \left( n + \frac{1}{2} \right) \hbar \omega \] so the possible transitions (from \(E_3\) to \(E_2\), \(E_2\) to \(E_1\), etc. are \(\Delta E = \hbar \omega\),
\[ 2\hbar \omega \text{, and } 3\hbar \omega \]. Specifically these calculations give:

\[ \hbar \omega = \hbar \sqrt{\frac{2k}{m}} = \left( 6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \right) \sqrt{\frac{2 \times \left( 1.1 \times 10^3 \text{ N/m} \right)}{1.673 \times 10^{-27} \text{ kg}}} = 0.755 \text{ eV} \text{ with a wavelength,} \]

\[ \lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.755 \text{ eV}} = 1640 \text{ nm} \]

\[ 2\hbar \omega = 2 \left( 6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \right) \sqrt{\frac{2 \times \left( 1.1 \times 10^3 \text{ N/m} \right)}{1.673 \times 10^{-27} \text{ kg}}} = 1.51 \text{ eV} \]

\[ \lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.51 \text{ eV}} = 821 \text{ nm} \]

\[ 3\hbar \omega = 3 \left( 6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \right) \sqrt{\frac{2 \times \left( 1.1 \times 10^3 \text{ N/m} \right)}{1.673 \times 10^{-27} \text{ kg}}} = 2.26 \text{ eV} \]

\[ \lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.26 \text{ eV}} = 549 \text{ nm} \]

44. In general for \(E > V_0\) we have \(R = 1 - T = 1 - \left[ 1 + \frac{V_0^2 \sin^2 (k_z L)}{4E(E - V_0)} \right]^{-1} \). If \(E \gg V_0\) then

\[ 4E(E - V_0) \approx 4E^2 \]. From the binomial theorem \((1 + x)^{-1} \approx 1 - x\) for small \(x\) and

\[ R \approx 1 - \left[ 1 - \frac{V_0^2 \sin^2 (k_z L)}{4E^2} \right] = \frac{V_0^2 \sin^2 (k_z L)}{4E^2} \]

which can be written as \(R \approx \left( \frac{V_0 \sin (k_z L)}{2E} \right)^2\).
46. \[ \kappa = \sqrt{\frac{2mc^2 (V_0 - E)}{\hbar c}} = \sqrt{\frac{2 \left( 511 \times 10^3 \text{ eV} \right) \left( 1.5 \text{ eV} \right)}{197.4 \text{ eV} \cdot \text{nm}}} = 6.27 \text{ nm}^{-1} \]

With a probability of \(2 \times 10^{-4}\) we know \(\kappa L \gg 1\) and we can use

\[ T = 16 \frac{E}{V_0} \left( 1 - \frac{E}{V_0} \right) e^{-2\kappa L} = 16 \frac{1}{2.5} \left( 1 - \frac{1}{2.5} \right) e^{-2\kappa L} = 3.84 e^{-2\kappa L} = 2 \times 10^{-4}. \]

Solving for \(L\):

\[ L = \frac{\ln \left( 1.92 \times 10^4 \right)}{2 \left( 6.27 \times 10^9 \text{ m}^{-1} \right)} = 7.86 \times 10^{-10} \text{ m}. \]

Now using the proton mass:

\[ \kappa = \sqrt{\frac{2mc^2 (V_0 - E)}{\hbar c}} = \sqrt{\frac{2 \left( 938.27 \times 10^6 \text{ eV} \right) \left( 1.5 \text{ eV} \right)}{197.4 \text{ eV} \cdot \text{nm}}} = 268.8 \text{ nm}^{-1}. \]

\[ T = 3.84 e^{-2\kappa L} = 3.84 e^{-2 \left( 268.8 \times 10^9 \text{ m}^{-1} \right) \left( 7.86 \times 10^{-10} \text{ m} \right)} = 1.2 \times 10^{-183}. \]

The proton's probability is much lower!

50. (a) We apply the time-dependent Schrödinger equation [Equation (6.1)] with \(V = 0\). The left side is \(i\hbar \frac{\partial \Psi}{\partial t} = i\hbar \left[ -\omega A \cos (kx - \omega t) + \omega B \sin (kx - \omega t) \right]\). The right side is

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = -\frac{\hbar^2}{2m} \left[ -k^2 A \sin (kx - \omega t) - k^2 B \cos (kx - \omega t) \right] = \frac{\hbar^2 k^2}{2m} \Psi. \]

Because of the complex number \(i\) and the extra minus sign on the left side, there is no way the two expressions can be equal. This is not a valid wave function.

(b) Using the Euler relation \(e^{i\theta} = \cos \theta + i \sin \theta\), this wave function is equivalent to \(\Psi(x, t) = A \cos (kx - \omega t) - iA \sin (kx - \omega t) = Ae^{i(kx - \omega t)}\). Example 6.2 verified that this is a valid wave function. You can also substitute the original wave function (containing sine and cosine) into Equation (6.1) to prove that it is valid.
53. (a) In general inside the box we have a superposition of sine and cosine functions, but only the sine function satisfies the boundary condition \( \psi(0) = 0 \), and thus \( \psi = A \sin(kx) \).

With \( V = 0 \) inside the well, \( E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \) or \( k = \sqrt{\frac{2mE}{\hbar}} \). Outside the well the decaying exponential is required as explained in section 6.4 of the text, with \( E = \frac{\hbar^2 k^2}{2m} + V_o \) which reduces to \( \kappa = ik = \frac{\sqrt{2m(V_o - E)}}{\hbar} \).

(b) Equating the wavefunctions and first derivatives at \( x = L \):
\[
A \sin(kL) = Be^{\kappa L} \quad \text{and} \quad kA \cos(kL) = -\kappa Be^{\kappa L}.
\]
Dividing the first equation by the second gives \( \frac{\tan(kL)}{\kappa} = -\frac{1}{\kappa} \) or \( \kappa \tan(kL) = -k \).

62. Using the known functions for \( \psi_1 \) and \( \psi_2 \) we see:
\[
\psi = \frac{1}{2} \psi_1 + \sqrt{\frac{3}{2}} \psi_2 = \frac{1}{2} \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x}{L} \right) + \sqrt{\frac{3}{2}} \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi x}{L} \right)
\]
For normalization:
\[
\int_0^L \psi^* \psi \, dx = \int_0^L \left( \frac{1}{2L} \sin^2 \left( \frac{\pi x}{L} \right) + \frac{3}{2L} \sin^2 \left( \frac{2\pi x}{L} \right) \right) \, dx.
\]
The third term vanishes because of the orthogonality of the trigonometric functions, leaving
\[
\int_0^L \psi^* \psi \, dx = \frac{1}{2L} \int_0^L \sin^2 \left( \frac{\pi x}{L} \right) \, dx + \frac{3}{2L} \int_0^L \sin^2 \left( \frac{2\pi x}{L} \right) \, dx
\]
\[
= \frac{1}{2L} \left( \frac{L}{2} \right) + \frac{3}{2L} \left( \frac{L}{2} \right) = 1 \quad \text{as required.}
\]

67. This tunneling problem is similar to Example 6.14. Because of the information given in the problem we will use Equation (6.73) to approximate the transmission probability. We know \( \kappa = \frac{\sqrt{2m(V_o - E)}}{\hbar} = \frac{\sqrt{2mc^2(V_o - E)}}{\hbar c} \) so \( \kappa = \sqrt{\frac{2 \times (5.11 \times 10^4) \times (0.9 \text{ eV})}{197.33 \text{ eV} \cdot \text{nm}}} = 4.86 \text{ nm}^{-1} \).

Then from Equation (6.73), \( T \approx 2e^{-2\kappa L} = 2e^{-2(4.86 \text{ nm}^{-1}) \times (1.3 \text{ nm})} = 6.51 \times 10^{-6} \).