Solution

********** Multiple-choice problems **********

1. A nearby star is 4 light years away from the earth. If an astronaut travels from the earth to the star with a speed of 0.8 c, how much time (unit: years) will have elapsed on his clock at the end of the journey?
   a) 2.4  b) 5  c) 4  d) 3  e) 8.3

2. A Michelson interferometer operates with a laser of wavelength \( \lambda = 600 \text{ nm} \). One arm of the interferometer has a mirror at the end of a copper bar, and the light path goes out parallel to the bar and is reflected by the mirror back along the same path. As the bar is cooled from \( T = 300 \text{ K} \) to \( T = 77 \text{ K} \), a total of 125 dark fringes are seen to pass through the eyepiece. It is concluded that the copper bar has contracted by:
   a) \( 7.5 \times 10^{-5} \text{ m} \)
   b) \( 3.75 \times 10^{-5} \text{ m} \)
   c) \( 1.5 \times 10^{-4} \text{ m} \)
   d) \( 3 \times 10^{-4} \text{ m} \)
   e) \( 1.9 \times 10^{-5} \text{ m} \)

3. Two events occur 100 m apart with an intervening time interval of 0.60 \( \mu \text{s} \). The speed of a reference frame in which they occur at the same coordinate is:
   a) 0
   b) 0.25c
   c) 0.56c
   d) 1.1c
   e) 1.8c

4. A double-slit experiment of electron uses a slit spacing \( d \) and electrons with energy \( E \). Here \( E \ll m_e \), i.e. non-relativistic limit. On a screen located a long distance \( L \) away (\( L \gg d \)), the interference pattern shows that adjacent bright fringes are separated by a small distance \( y \). If the electron energy \( E \) is then doubled, what will be the new separation between adjacent bright fringes?
   a) \( 4y \)
   b) \( y/\sqrt{2} \)
   c) \( y/4 \)
   d) \( y/2 \)
   e) \( \sqrt{2}y \)
5. The threshold wavelength for photoemission in calcium is 384 nm. If the light of wavelength 200 nm is used, what will be the photoelectric stopping potential \( V_s \) in volts?

- \( V_s < 2.1 \)
- \( 2.1 \leq V_s < 3.1 \)
- \( 3.1 \leq V_s < 4.1 \)
- \( 4.1 \leq V_s < 5.1 \)
- \( V_s \geq 5.1 \)

\[
eV_2 = \frac{h \nu}{\lambda} - \phi \Rightarrow \frac{h \nu}{\lambda \hbar} - \phi = \frac{1240 \text{ nm - ev}}{384} = 3.23 \text{ (eV)}
\]

\[
eV_2 = \frac{h \nu}{\lambda} - \phi = \frac{1240}{200} - 3.23 = 6.2 - 3.23 = 2.97 \text{ (eV)}
\]

6. In a metal, at the absolute zero of temperature

- all motion ceases
- the Fermi energy is zero
- the Fermi speed is zero
- the average kinetic energy of the conduction electrons is zero
- the average kinetic energy of the conduction electrons differs significantly from zero

7. The figure shows the Fermi function for a solid at two different temperatures \( T_A \) and \( T_B \). From the shape of these curves we can tell that:

- \( T_A < T_B \)
- \( T_A > T_B \)
- \( T_B = 0 \text{ K} \)
- states above the Fermi energy are more likely to be occupied at \( T_A \) than at \( T_B \).
- the solid is an insulator.

8. A 1.00 g sample of pure KCl from the chemistry stockroom is found to be radioactive and to decay at an absolute rate \( R \) of 1600 counts/s. The decay is traced to the element potassium and in particular to the isotope \(^{40}\text{K}\), which constitutes 1.18% of normal potassium. The molecular weight of KCl is 74.9 g/mole. The disintegration constant \( \lambda \) is

- \( 1.12 \times 10^{-17} \text{s}^{-1} \)
- \( 1.69 \times 10^{-17} \text{s}^{-1} \)
- \( 1.96 \times 10^{-17} \text{s}^{-1} \)
- \( 2.30 \times 10^{-17} \text{s}^{-1} \)
- none of these

\[
R = \left| \frac{dN}{dt} \right| = 1600 \quad N = \frac{m}{m_k} \cdot N_A \cdot f
\]

\[
f = 1.18\%
\]

\[
\lambda = \frac{1}{N} \left| \frac{dN}{dt} \right| = \frac{1}{74.9} \times 6.02 \times 10^{23} \times 1.18\% = 9.8 \times 10^{19} \text{ s}^{-1}
\]
9. 50 neutrons with mass $m$ are placed in a one-dimensional square well, of size $l$. Neglect any interaction between neutrons. At zero temperature, the highest energy neutron will have an energy of: [hint: neutrons are fermions.]

- a) $625(h^2/2ml^2)$
- b) $(h^2/2ml^2)$
- c) $50(h^2/2ml^2)$
- d) $25(h^2/2ml^2)$
- e) $2500(h^2/2ml^2)$

10. Neutrinos
- a) are very high energy photons $\times$
- b) are thought to exist, but have not yet actually been detected $\times$
- c) carry little energy $\times$ can carry a lot of energy.
- d) interact weakly with matter $\checkmark$ through weak force
- e) are anti-neutrons $\times$

11. In a doped semiconductor material of the p-type $\underline{\text{holes dominate}}$

- a) the density of conduction electrons far exceeds the density of holes $\times$
- b) the holes carry negative charge $\underline{\text{positive}}$ $\times$
- c) the current is carried mainly by the electrons $\times$
- d) the current is carried mainly by the holes $\checkmark$
- e) the valence band is completely full $\times$ no holes in full band.

12. In a hydrogen atom, in the ground state, described by $\psi_1(r) = (\pi r_0^3)^{-1/2} e^{-r_0}$, approximately what is the ratio of the probability $P(r > r_0)$ that the electron will be found beyond the Bohr radius, $r_0$, to the probability $P(r \leq r_0)$ that it is found inside the distance?

- a) 1.0
- b) 0.5
- c) 2.1
- d) 1.5
- e) 2.5

13. Pions ($\pi^-$) have a half-life of $2.2 \times 10^{-8}$ sec as measured by an observer at rest with respect to them. An observer in a laboratory sees a beam of pions traveling at 0.995c. As the beam passes him he counts 1000 pions per second. How many pions per second will be left after the beam travels 10 m (measured by the observer in the laboratory) further?

- a) 360
- b) 510
- c) 630
- d) 900
- e) 860

\[ \text{In Lab frame} \]

\[ \lambda = \frac{1}{\sqrt{1 - v^2}} \approx 10 \Rightarrow \lambda_{L} = \lambda_{L0} = 2.2 \times 10^{-7} \text{(sec)} \]

\[ \Delta t = \frac{\Delta x}{v} = \frac{10}{0.995 \times 3 \times 10^8} = 3.35 \times 10^{-8} \text{(sec)} \]

\[ N(t) = N_0 e^{-\frac{t}{\lambda}} \]
14. In an atom with two electrons on \( n = 3 \) level, one electron on \( p \) orbital and the other is on \( d \) orbital. Which of the following spectroscopy symbols \( (n^{2s+1}L_J) \) is NOT a possible state?

a) \( 3^1P_1 \)  
b) \( 3^3D_3 \)  
c) \( 3^3F_2 \)  
d) \( 3^3P_1 \)  
e) \( 3^1S_0 \)  
\( L=0 \)

15. Which of the following is not true?

a) The rest mass of a hydrogen atom is less than the sum of the rest masses of an electron and a proton. \( T \Delta m = 13.6 \text{ eV} \)

b) The rest mass of a helium nucleus is less than the sum of the rest masses of two protons and two neutrons. \( T \Delta m = \text{Binding Energy} \)

c) The rest mass of an iron nucleus \( (^{56}_{26}\text{Fe}) \) is less than the sum of the rest mass of \( 13 \) \( \alpha \) particles and \( 4 \) neutrons. \( T \Delta m = \text{Binding Energy} \)

d) The rest mass of a uranium nucleus \( (^{235}_{92}\text{U}) \) is less than the sum of the rest masses of \( ^{114}_{44}\text{Ru}, ^{128}_{50}\text{Cd} \), and \( 3 \) neutrons. \( T \Delta m = \text{Binding Energy} \)

e) The rest mass of a neutron is more than the sum of the rest masses of a proton and an electron. \( n \rightarrow p^+ + e^- + \bar{\nu}_e \)

16. For a semiconductor with a band gap \( E_g = 0.4 \text{ eV} \) and the Fermi level in the middle of the gap, by how many times does the occupancy of a state at the very bottom of the conduction band increase when temperature is raised from 300 to 600 K?

\[
\int (\varepsilon) = \frac{1}{e^{\frac{E_g}{kT}} + 1} \quad \Rightarrow \quad \frac{f(E)}{f(E^b)} = e^{-\beta E_g/2}
\]

\[\begin{align*}
\text{a)} & \quad 47 \\
\text{b)} & \quad 0.000435 \\
\text{c)} & \quad 0.02 \\
\text{d)} & \quad 2 \\
\text{e)} & \quad 15.38
\end{align*}\]

\[
\frac{T_k}{T_e} = \frac{600}{300} = 2 \Rightarrow 4 = e^{-\beta E_g/2} = e^{\frac{0.4}{4 \times 0.026} \times 2} \approx 4.7
\]

17. The ionization energy of hydrogen is 13.6 eV. What would the ionization energy be if the electron were replaced by a muon \( (\mu^-) \)? Mass of muon is \( m_\mu \approx 105.658 \text{ MeV/c}^2 \), and mass of electron is \( m_e \approx 0.511 \text{ MeV/c}^2 \)

\[\begin{align*}
\text{a)} & \quad 0.066 \text{ eV} \\
\text{b)} & \quad 13.6 \text{ eV} \\
\text{c)} & \quad 2.82 \text{ keV} \\
\text{d)} & \quad 196 \text{ eV} \\
\text{e)} & \quad 0.94 \text{ eV}
\end{align*}\]

\[\begin{align*}
E_{\mu}^e &= -\frac{m_e^2 c^2}{2n^2} \times \frac{m_\mu c^2}{m_e} \\
E_{\mu}^e &= \frac{m_\mu c^2}{m_e} \times \frac{105.658 \text{ MeV}}{0.511} \times 13.6 = 2.82 \times 10^3 (\text{eV})
\end{align*}\]
18. Which of the following are acceptable descriptions of a state of Hydrogen atom?

I \( n = 2, l = 1, m_l = 1, s = 1, m_s = 0 \) \( \times \)
II \( n = 2, l = 0, m_l = 0, s = 1/2, j = 3/2, m_j = -3/2 \) \( \times \)
III \( n = 2, l = 1, s = 1/2, j = 3/2, m_j = -3/2 \) \( \phi \)
IV \( n = 2, l = 1, m_l = -1, s = 1/2, m_s = -1/2 \) \( \checkmark \)

- a) All
- b) I and III
- c) II and III
- d) III and IV
- e) II and IV

19. Only one of the following reactions or decays can occur. Which one? In case of reactions, sufficient kinetic energy is available in the initial state.

a) \( \pi^- + p \to n + \bar{n} \times \) *Baryon # not conserved*

b) \( p + p \to p + n + \pi^0 \times \) *Charge not conserved*

c) \( p + p \to \pi^+ + \pi^+ \times \) *Baryon # not conserved*

d) \( p + \bar{p} \to \pi^+ + \pi^- \checkmark \) 1+1 \( \to \) 0+0

e) \( e^- \to \pi^- + \nu_e + \bar{\nu}_e \times \) *Electron does decay to \( e^- \).*

20. The energy of the \( n = 2 \) level of hydrogen is 10 eV above the \( n = 1 \) ground state. At 11,600 K \( (k_B T = 1 \text{ eV}) \) what fraction of hydrogen atoms are in an \( n = 2 \) state? [Note: think carefully here]

a) \( 1 \times e^{-10} \)

b) \( 2 \times e^{-10} \)

c) \( 3 \times e^{-10} \)

d) \( 4 \times e^{-10} \)

\[ e^{-8} \Rightarrow \frac{n_2}{n_1} = \frac{\Phi(2)}{\Phi(1)} = \frac{8 \cdot e^{-\beta E_2}}{2 \cdot e^{-\beta E_1}} = 4 \cdot e^{-4 \beta E} \approx 4 \cdot e^{-10} \]

*************** End of multiple-choice problems ***************

\( k_B T = 1 \text{ eV} \Rightarrow \beta = 10 \)
open-ended problems

21. The bond length of N\textsubscript{2} molecules is approximately 0.11 nm. The effective spring constant of the N-N covalent bond is approximately \( k = 2247 \text{ N/m} \). Mass of nitrogen atom is \( m_N = 14 \text{ u} \).

(5 points) (a) Use Maxwell-Boltzmann factors to find the population ratio \( P_1/P_0 \) of the \( \ell = 0 \) and \( \ell = 1 \) states of rotational motion of N\textsubscript{2} molecules at room temperature (300 K). Moment of inertia of diatomic molecules is \( I = \mu R^2 \), where \( \mu \) is the reduced mass and \( R \) is the bond length.

(5 points) (b) Use Maxwell-Boltzmann factors to find the population ratio \( P_1/P_0 \) of the \( n = 0 \) and \( n = 1 \) states of vibrational motion of N\textsubscript{2} at room temperature (300 K).

Sol:

(a) Boltzmann factor: \( P(E) \propto e^{-\beta E} \Rightarrow \frac{P_1}{P_0} = \exp[-\beta(E_1 - E_0)]. \)

\[
E_\ell = \frac{\hbar^2}{2I} \ell(\ell + 1) \Rightarrow E_1 - E_0 = \frac{\hbar^2}{2I} \cdot 2 = \frac{\hbar^2}{I}.
\]

The effective mass of a N\textsubscript{2} molecule is:

\[
\mu = \frac{m_N}{2} = \frac{14 \times 1.66 \times 10^{-27}}{2} = 1.16 \times 10^{-26} \text{ kg},
\]

So the moment of inertia is:

\[
I = \mu R^2 = \frac{m_N R^2}{2} = 1.16 \times 10^{-26} \times (0.11 \times 10^{-9})^2 = 1.4 \times 10^{-46} \text{ (kg}\cdot\text{m}^2).\]

\[
\Rightarrow E_1 - E_0 = \frac{(1.05 \times 10^{-34})^2}{1.4 \times 10^{-46}} = 7.875 \times 10^{-23} \text{ (J)}.
\]

For room temperature (300 K), \( k_B T = 1.38 \times 10^{-23} \cdot 300 = 4.14 \times 10^{-21} \text{ J} \),

\[
\Rightarrow \frac{P_1}{P_0} = \exp\left(\frac{-7.875 \times 10^{-23}}{4.14}\right) = 0.98.
\]

(b) \( E_{\text{vib}} = \hbar \omega \left(n + \frac{1}{2}\right), \) where

\[
\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{2247}{1.16 \times 10^{-26}}} = 4.4 \times 10^{14} \text{ Hz}^{-1}
\]

\[
\Rightarrow \hbar \omega = 4.62 \times 10^{-20} \text{ J}.
\]

\[
\Rightarrow \frac{P_1}{P_0} = \exp[-\beta(E_1 - E_0)] = \exp(-\beta \hbar \omega) = \exp\left(-\frac{46.2}{4.14}\right) = 1.42 \times 10^{-5}.
\]
22. One way to decide whether Maxwell-Boltzmann statistics are valid for an ideal gas is to compare the de Broglie wavelength $\lambda$ of a typical molecule with the average inter-molecule spacing $d$. If $\lambda \geq d$ then Maxwell-Boltzmann statistics are not valid.

(3 points) (a) Using RMS speed $v_{\text{rms}}$ to show that:

$$\lambda = \frac{h}{\sqrt{3mk_B T}}$$

Here $m$ is the mass of the molecule and $T$ is the gas temperature.

(3 points) (b) Use the fact that $V/N = d^3$ to show that the equality $\lambda = d$ condition can be expressed as:

$$\frac{N}{V} \frac{h^3}{(3mk_B T)^{3/2}} = 1$$

(4 points) (c) Use above results to determine the critical temperature $T_c$ of helium gas when Maxwell-Boltzmann statistics are not valid. Assume the gas density is fixed at the standard condition value (i.e., 22.4 liters for 1 mole of gas).

Sol:

(a) The de Broglie wavelength is $\lambda = \frac{h}{P} = \frac{h}{m v_{\text{rms}}} = \frac{h}{m \sqrt{\frac{3k_B T}{m}}} = \frac{h}{\sqrt{3mk_B T}}$.

(b) $V/N = d^3$ and $\lambda = d$ $\Rightarrow$ $V = \lambda^3 = \frac{h^3}{(3mk_B T)^{3/2}}$.

$\Rightarrow$ $\frac{N}{V} \frac{h^3}{(3mk_B T)^{3/2}} = 1$.

(c) The equation in (b) can be rewritten as:

$$T_c = \left( \frac{N}{V} \right)^{\frac{2}{3}} \frac{h^2}{3mk_B} = \left( \frac{6.02 \times 10^{23}}{22.4 \times 10^{-3}} \right)^{\frac{2}{3}} \frac{(6.63 \times 10^{-34})^2}{3 \times 4 \times 10^{-27} \times 1.38 \times 10^{-23}}$$

$= 8.97 \times 10^{16} \times 43.9569 \times 10^{-68}$

$\Rightarrow T_c = 0.143 \text{K}$
23. Low energy nuclear reaction kinematics, i.e. kinetic energies are typically much lower than the rest energies. Consider the reaction $x + X \rightarrow y + Y$, where $X$ is at rest and $x$ is the projectile with speed $v_x$. Define energy release $Q$ as: $Q = (M_x + M_X - M_y - M_Y)c^2$. For exothermic ($Q < 0$) reaction, show that the minimum kinetic energy $K_{th} = \frac{1}{2}m_xv_x^2$ needed to initiate the reaction is:

$$K_{th} = -Q \left( \frac{M_x + M_X}{M_X} \right)$$

Sol:

For the center of mass (CM) frame,

$$(M_x + M_X)v_{cm} = M_xv_x \quad \Rightarrow \quad v_{cm} = \frac{M_x}{M_x + M_X}v_x.$$ 

In the CM frame (the prime frame), velocity of $M_x$ and $M_X$ are:

$$v'_x = v_x - v_{cm} = \frac{M_X}{M_x + M_X}v_x, \quad v'_X = v_{cm} = \frac{M_x}{M_x + M_X}v_x.$$ 

Energy conservation in the CM frame is:

$$\frac{1}{2}M_xv_x'^2 + \frac{1}{2}M_Xv'_X^2 + (M_x + M_X)c^2 = \frac{1}{2}M_yv_y'^2 + \frac{1}{2}M_Yv'_Y^2 + (M_y + M_Y)c^2$$

For the minimum kinetic energy need to initiate the reaction, $v'_Y$ and $v_y$ must equal to zero in the CM frame, i.e., $v'_Y = v'_y = 0$. Note here we apply the low energy approximation, namely, the energy release $Q$ is much less than rest masses so that $M_x + M_X \approx M_y + M_Y$ is applied in momentum conservation. Thus,

$$\frac{1}{2}M_xv_x'^2 + \frac{1}{2}M_Xv'_X^2 = -Q$$

$$\Rightarrow \frac{1}{2}M_x \left( \frac{M_X}{M_x + M_X}v_x \right)^2 + \frac{1}{2}M_X \left( \frac{M_x}{M_x + M_X}v_x \right)^2 = -Q$$

$$\Rightarrow \frac{1}{2}M_xv_x^2 \cdot M_X \cdot \left[ \frac{M_X}{(M_x + M_X)^2} + \frac{M_x}{(M_x + M_X)^2} \right] = -Q$$

$$\Rightarrow \frac{1}{2}M_xv_x^2 \cdot M_X \cdot \left[ \frac{M_X + M_x}{(M_x + M_X)^2} \right] = -Q$$

$$\Rightarrow K_{th} = \frac{1}{2}M_xv_x^2 = -Q \cdot \left( \frac{M_x + M_X}{M_X} \right).$$
24. Considering a monoatomic ideal gas in 2 dimension (2D).

(a) Use Boltzmann factor to derive the velocity distribution function \( P(v_x, v_y) \) of the 2D ideal gas with proper normalization, i.e.

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(v_x, v_y) dv_x dv_y = 1
\]

[hint: start from 1D velocity distribution.]

(b) Use the result of (a) to derive the speed distribution function \( P(v) \) (where \( v = \sqrt{v_x^2 + v_y^2} \)) of the 2D ideal gas.

Sol:

(a) Using Boltzmann factor, the probability of finding a gas molecule at \( v_x \) for the projection of motion on \( x \)-axis is:

\[ P(v_x) \propto e^{-\beta \frac{1}{2} mv_x^2}, \]

where \( \beta = \left( \frac{k_B T}{m} \right)^{-1} \). Normalization requires:

\[ \int_{-\infty}^{+\infty} P(v_x) dv_x = 1, \]

\[ \Rightarrow \int_{-\infty}^{+\infty} C \cdot e^{-\beta \frac{1}{2} mv_x^2} dv_x = 1. \]

Let \( t = \sqrt{\frac{m \beta}{2}} \),

\[ \Rightarrow 2C \cdot \frac{1}{m \beta} \cdot \frac{\sqrt{\pi}}{2} = 1 \Rightarrow C = \sqrt{\frac{m \beta}{2 \pi}}. \]

\[ \Rightarrow P(v_x) = \sqrt{\frac{m \beta}{2 \pi}} e^{-\beta \frac{1}{2} mv_x^2}. \]

Similarly, we can derive: \( P(v_y) \) by replacing \( x \) with \( y \):

\[ P(v_y) = \sqrt{\frac{m \beta}{2 \pi}} e^{-\beta \frac{1}{2} mv_y^2}, \]

and it satisfies normalization condition \( \int_{-\infty}^{+\infty} P(v_y) dv_y = 1 \). Because \( x \) and \( y \) are orthogonal to each other, i.e., they are independent of each others.

Therefore:

\[ P(v_x, v_y) = P(v_x)P(v_y) = \frac{m \beta}{2 \pi} \cdot e^{-\beta \frac{1}{2} (v_x^2 + v_y^2)}. \]

(b) Because \( v = \sqrt{v_x^2 + v_y^2} \), we need to integrate over a circle radius of \( v \) to get the the speed distribution function \( P(v) \):

\[ P(v) = \int_v P(v_x, v_y) dv_x dv_y = \int_0^{2\pi} \frac{m \beta}{2 \pi} \cdot e^{-\beta \frac{1}{2} v^2} v d\phi = m \beta \cdot v e^{-\beta \frac{1}{2} v^2}. \]

Normalization:

\[ \int_0^{\infty} P(v) dv = \int_0^{\infty} m \beta ve^{-\beta \frac{1}{2} v^2} dv = \int_0^{\infty} e^{-t^2} d(t^2) = 1, \]

where \( t = \sqrt{\frac{m \beta}{2}}. \)