Lecture 20

Relativistic Energy and Momentum

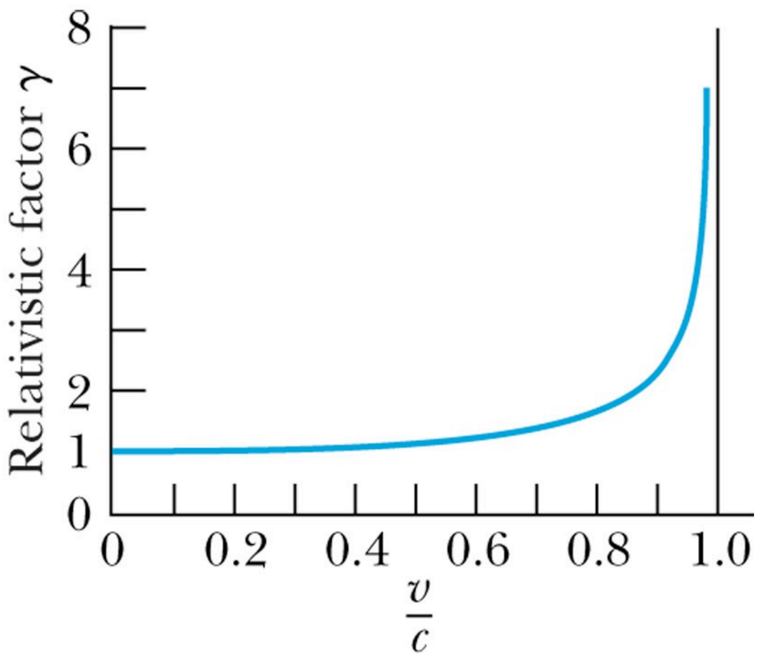
Nov. 14, 2007

The Gamma Factor

The factor $\frac{1}{\sqrt{1 - v^2/c^2}}$ occurs all the time in relativity.

We call it γ $$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

We then have: $E = m\gamma c^2$ $\vec{p} = m\gamma\vec{v}$



$\gamma \to \infty$ as $v \to c$

$\Rightarrow$ E and $\vec{p}$ $\to \infty$ as $v \to c$

Compton Scattering

It's now easy to derive the Compton scattering formula

$$\mathbb{P}_\gamma + \mathbb{P}_e \; = \; \mathbb{P}'_e + \mathbb{P}'_\gamma$$

$$\mathbb{P}_\gamma + \mathbb{P}_e - \mathbb{P}'_\gamma \; = \; \mathbb{P}'_e$$

$$(\mathbb{P}_\gamma + \mathbb{P}_e - \mathbb{P}'_\gamma)^2 \; = \; (\mathbb{P}'_e)^2$$

$$\mathbb{P}_\gamma^2 + \mathbb{P}_e^2 + \mathbb{P}'^2_\gamma + 2\,\mathbb{P}_\gamma \cdot \mathbb{P}_e - 2\,\mathbb{P}_\gamma \cdot \mathbb{P}'_\gamma - 2\,\mathbb{P}_e \cdot \mathbb{P}'_\gamma = \mathbb{P}'^2_e$$

$$m_e^2 c^2 + 2\left(\frac{E_\gamma m_e c^2}{c^2} - \frac{E_\gamma E'_\gamma}{c^2}(1-\cos\theta) - \frac{E'_\gamma m_e c^2}{c^2}\right) = m_e^2 c^2$$

$$E_\gamma m_e c^2 - E'_\gamma m_e c^2 = E_\gamma E'_\gamma (1-\cos\theta)$$

Compton Scattering

$$\frac{E_\gamma - E'_\gamma}{E_\gamma E'_\gamma} = \frac{1}{m_e c^2}\,(1 - \cos\theta)$$

$$\frac{f - f'}{f f'} = \frac{h}{m_e c^2}\,(1 - \cos\theta)$$

$$\frac{1}{f'} - \frac{1}{f} = \frac{h}{m_e c^2}\,(1 - \cos\theta)$$

$$\lambda' - \lambda = \frac{h}{m_e c}\,(1 - \cos\theta)$$

Invariant Mass

Decay of unstable particle of mass M at rest

$$\mathbb{P}_1 = (E_1/c, \vec{p}_1) \qquad \mathbb{P}_2 = (E_2/c, \vec{p}_2)$$

Conservation of momentum: $\vec{p}_2 = -\vec{p}_1$

Conservation of energy: $E_1 + E_2 = Mc^2$

$$(\mathbb{P}_1 + \mathbb{P}_2)^2 = \left(\frac{E_1 + E_2}{c}\right)^2 - |\vec{p}_1 + \vec{p}_2|^2 = \left(\frac{Mc^2}{c}\right)^2 = M^2c^2$$

This is invariant (the same) in all inertial frames

Invariant Mass

In the lab frame we may produce particle M in an high energy collision. It then decays to particles 1 and 2 while moving. With our detector we measure the energy of momentum (the four-momentum vector) of particles 1 and 2. How can we tell that they came from the decay of a parent particle of mass M?

$$(\mathbb{P}_1 + \mathbb{P}_2)^2 = \mathbb{P}_1^2 + \mathbb{P}_2^2 + 2\,\mathbb{P}_1 \cdot \mathbb{P}_2$$

$$= m_1^2 c^2 + m_2^2 c^2 + 2\,E_1 E_2 / c^2 - 2\,\vec{p_1} \cdot \vec{p_2}$$

$$= m_1^2 c^2 + m_2^2 c^2 + 2\,E_1 E_2 / c^2 - 2\,|\vec{p_1}||\vec{p_2}| \cos\theta = M^2 c^2$$

where θ is the angle between the directions of particle 1 and 2.

Ultra relativistic Case

In ultrarelativistic case this simplifies

$$E_1 \gg m_1 c^2 \qquad E_2 \gg m_2 c^2$$

$$\Rightarrow \qquad E_1 \approx |p_1| c \qquad E_2 \approx |p_2| c$$

$$(\mathbb{P}_1 + \mathbb{P}_2)^2 = \mathbb{P}_1^2 + \mathbb{P}_2^2 + 2\,\mathbb{P}_1 \cdot \mathbb{P}_2$$

$$= m_1^2 c^2 + m_2^2 c^2 + 2\,E_1 E_2 / c^2 - 2\,|\vec{p_1}||\vec{p_2}| \cos\theta$$

$$\approx \frac{2E_1 E_2}{c^2} (1 - \cos\theta)$$

$$2E_1 E_2 (1 - \cos\theta) = M^2 c^4$$