Lecture 19

Special Relativity

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Special Relativity

We'll now begin our study of special relativity, the other pillar of modern physics.

I won't assume that you already know anything about relativity but I'm not going to cover the Michelson-Morley null result experiment which provided the motivation for relativity. I'm also not going to cover the peculiar and interesting properties that result from relativity:

- time dilation
- length contraction
- breakdown of simultaneity

If you need to review those or haven't had relativity before you should read over Chapter 2 in the text but in general we won't need this for our discussion

We'll focus on the study of 4-momenta and energy-momentum relations based on material in Sections 3-3 and 3-4 in the text.

Three Dimensional Space

Before we begin to discuss relativistic space-time, lets review some properties and concepts of ordinary three-dimensional space. We'll then carry over these ideas to four-dimensional space-time.

These ideas include:

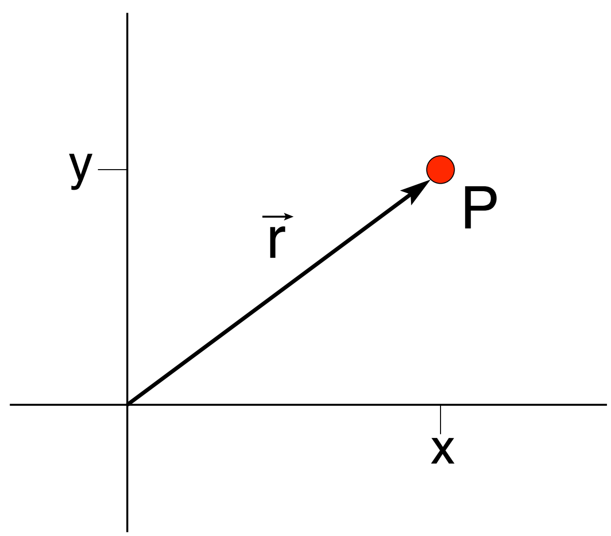
- vectors
- distance between two points
- invariance under a transformation
- dot product of two vectors

Vectors

The position in three-dimensional space is completely determined by specifying three numbers: the x, y and z coordinates of the point with respect to a coordinate system.

These three numbers can be formulated as a vector that gives the distance and direction of the point from the origin of the coordinate system.

$$\vec{r} = (x, y, z)$$

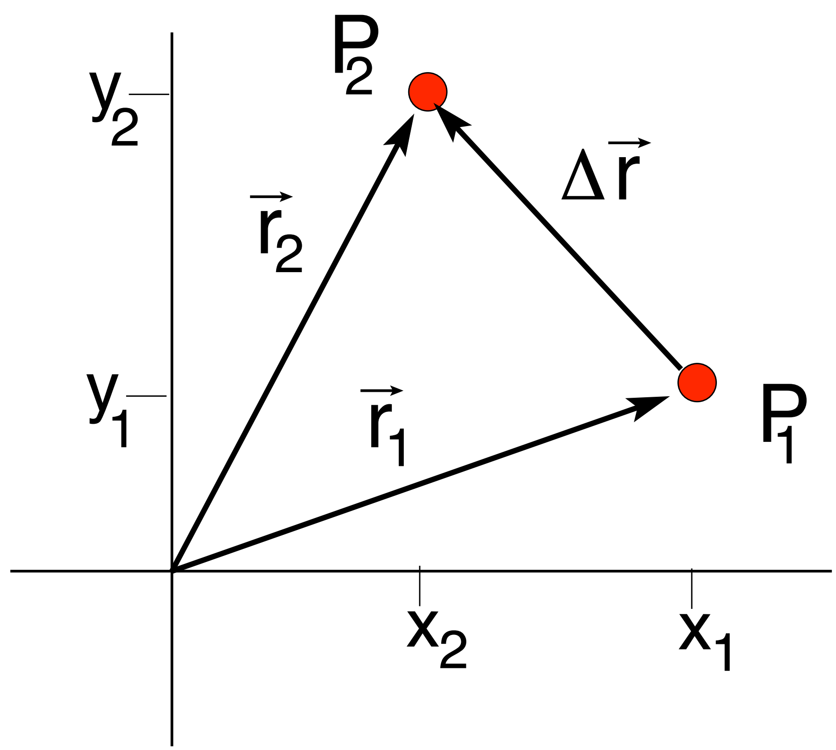


Separation between Two Points

The separation between two points is also given by a vector.

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$= (x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1) = (\Delta x,\, \Delta y,\, \Delta z)$$



Distance between Two Points

Since our space is Euclidean, the Pythagorean Theorem holds so that the square of the distance between two points is given by :

$$(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$(\Delta r)^2 = \sum_{i,j=1}^{3} \delta_{ij} \Delta r_i \Delta r_j$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is called the metric of our space.

it specifies a three-dimensional Euclidean space.

Invariance

Since our Euclidean space is isotropic, the distance between two points is invariant (doesn't change) if the system is rotated.

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$$

$$(\Delta r)^2 = (\Delta r')^2$$

where the primed quantities are the coordinates of the rotated system.

Rotational Transformation

In order for:

$$(\Delta r)^2 = (\Delta r')^2$$

to hold, the components of $\vec{r}$ must transform in a specific way under rotations.

For example, under rotations about the z-axis:

$$\Delta x' = \cos\theta\,\Delta x + \sin\theta\,\Delta y$$

$$\Delta y' = -\sin\theta\,\Delta x + \cos\theta\,\Delta y$$

$$\Delta z' = \Delta z$$

with $\qquad \cos^2\theta + \sin^2\theta = 1$

Dot Product

Define the dot product of two vectors as:

$$\Delta\vec{r}\cdot\Delta\vec{r} \equiv (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta r)^2$$

The fact that this dot product is invariant under rotations depends only on the way that the vector $\Delta\vec{r}$ transforms under rotation.

Any other quantities that have three components and that transform like the components of $\Delta\vec{r}$ under rotation will also form a dot product that is invariant.

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

is invariant under rotations if $\vec{A}$ and $\vec{B}$ transform like vectors.

Scalar Multiplication

How can we make additional vectors? Easy multiply by a scalar a number that doesn't change under the transformation.

If α is a scalar and $\vec{A}$ is a vector, then $\alpha\vec{A}$ is also a vector.

The reason is because the rotational transformations are linear.

$$\alpha A'_x = \cos\theta\,\alpha A_x + \sin\theta\,\alpha A_y$$

$$\alpha A'_y = -\sin\theta\,\alpha A_x + \cos\theta\,\alpha A_y$$

Event

An event is something that happens at a certain point in space at a certain time.

In order to specify an event we must give four numbers: the three spatial coordinates and the time.

Two events are separated in both time: Δt

and in space: $\Delta \vec{r}$

For example: for a moving particle we might have

Event 1: particle at $\vec{r}_1$ at time t_1

Event 2: particle at $\vec{r}_2$ at time t_2

Velocity

Δt doesn't change under rotations $\quad \Delta t = \Delta t'$

$\Rightarrow \quad \Delta t$ is a scalar under rotations

We can form additional vector by multiplying the vector $\Delta\vec{r}$ by the scalar $1/\Delta t$.

velocity: $$\vec{v} = \frac{\Delta\vec{r}}{\Delta t}$$

We can make other vectors by scalar multiplication

acceleration: $$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

momentum: $$\vec{p} = m\vec{v}$$

Uniform Velocity Boost

Now consider the transformation from one coordinate system to the other by a uniform velocity boost. The observer in one frame is moving with a constant velocity with respect to an observer in the other frame. In the classical view, the spatial separation of two events is not invariant under this transformation.

If the primed frame is moving with speed v in the $+x$ direction with respect to the unprimed frame:

$$\Delta x' = \Delta x - v\Delta t$$

$$\Delta y' = \Delta y$$

$$\Delta z' = \Delta z$$

and $$\Delta t' = \Delta t$$

But this isn't correct!

Relativistic Space-Time

In relativity, space and time are not separate entities but form a four-dimensional space-time.

Under Lorentz (uniform velocity) boosts space and time transform into one another.

In space-time, the space and time separation between two events are combined to give an overall separation.

$$(\Delta R)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

4-Vectors

The location of an event in space-time can be specified by a four-component vector.

$$\mathbb{R} = (ct,\, x,\, y,\, z)$$

The square of the separation between two events is given by:

$$(\Delta R)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Minkowski Space

Space time is a Minkowski not a Euclidean space.

The metric is :

$$(\Delta R)^2 = \sum_{\mu,\nu=1}^{4} \delta_{\mu\nu} \Delta r_\mu \Delta r_\nu$$

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Four dimensional space-time consists of three space dimensions and one time dimension. The minus sign is what distinguishes time from space. If the metric were:

$\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ it would be just four Euclidean space dimensions

The Special Theory of Relativity

The square of the distance between two events

$$(\Delta R)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

is invariant under rotations and Lorentz boosts.

That's it. That's all there is to the theory of relativity.

c is the fundamental constant of the theory. It has units of velocity. It multiples the time component so that it has the same units as the space components.

Invariance

$(\Delta R)^2$ is clearly invariant under rotations since

$$(\Delta R)^2 = (c\Delta t)^2 - (\Delta r)^2$$

and $\quad (\Delta r')^2 = (\Delta r)^2 \qquad \Delta t' = \Delta t \qquad$ under rotations

In order for $(\Delta R)^2$ to be invariant under Lorentz boosts the four-vector $\Delta\mathbb{R}$ must transform in a certain way.

Under a velocity boost of v in the positive x direction.

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}}$$

$$\Delta y' = \Delta y$$

$$\Delta z' = \Delta z$$

$$\Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}$$

Lorentz Transformations

Under a velocity boost of v in the positive x direction.

$$\begin{aligned}
\Delta x' &= \frac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}} \\
\Delta y' &= \Delta y \\
\Delta z' &= \Delta z \\
\Delta t' &= \frac{\Delta t - v\Delta x/c^2}{\sqrt{1 - v^2/c^2}}
\end{aligned}$$

alternatively:

$$\begin{aligned}
\Delta x' &= \cosh\vartheta\,\Delta x - \sinh\vartheta\,\Delta t \\
\Delta y' &= \Delta y \\
\Delta z' &= \Delta z \\
\Delta t' &= -\sinh\vartheta\,\Delta x + \cosh\vartheta\,\Delta t
\end{aligned}$$

with $\tanh\vartheta = v/c$

Four-Vector Dot Product

Define the dot product of two four-vectors as:

$$\Delta\mathbb{R}\cdot\Delta\mathbb{R} \equiv (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta\mathbb{R})^2$$

The fact that this dot product is invariant under rotations depends only on the way that the vector $\Delta\mathbb{R}$ transforms under rotation.

Any other quantities that have four components and that transform like the components of $\Delta\mathbb{R}$ will also form a dot product that is invariant.

$$\mathbb{A}\cdot\mathbb{B} = A_t B_t - A_x B_x - A_y B_y - A_z B_z$$

is invariant under rotations if $\mathbb{A}$ and $\mathbb{B}$ transform like four-vectors.

Proper Time

Now let's make new four-vectors by multiplying by a scalar.

Can we form a velocity four-vector by multiplying by $1/\Delta t$ as we did before?

No. $\Delta\mathbb{R}/\Delta t$ is not a four-vector. Because Δt is not a scalar. It is different for different reference frames $\Delta t' \neq \Delta t$.

$$\text{Define} \qquad (c\,\Delta\tau)^2 \equiv (\Delta R)^2 = (c\,\Delta t)^2 - (\Delta r)^2$$

$$\Rightarrow \qquad c\,\Delta\tau = \sqrt{(c\Delta t)^2 - (\Delta r)^2} = c\,\Delta t\sqrt{1 - v^2/c^2}$$

is an invariant (scalar). It is the same in all reference frames

$\Delta\tau$ is called the proper time. It is the time difference in the frame where $\Delta r = 0$, that is, the frame in which the two events occur at the same location.

Velocity Four-Vector

We can now form a velocity four-vector.

$$\mathbb{V} = \frac{\Delta\mathbb{R}}{\Delta\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{\Delta\mathbb{R}}{\Delta t}$$

$$= \frac{1}{\sqrt{1 - v^2/c^2}} \left(\frac{c\,\Delta t}{\Delta t}, \frac{\Delta\vec{r}}{\Delta t} \right) = \frac{1}{\sqrt{1 - v^2/c^2}} (c, \vec{v})$$

What is the invariant $\mathbb{V} \cdot \mathbb{V}$?

$$\mathbb{V} \cdot \mathbb{V} = \frac{1}{1 - v^2/c^2} (c^2 - \vec{v} \cdot \vec{v}) = \frac{c^2(1 - v^2/c^2)}{1 - v^2/c^2} = c^2$$

$\mathbb{V} \cdot \mathbb{V}$ is equal to c^2 for all velocities.

Monentum Four-Vector

Now form the relativistic four-momentum of the particle by multiplying $\mathbb{V}$ by the particle's mass.

$$\mathbb{P} = m\mathbb{V} = \frac{1}{\sqrt{1 - v^2/c^2}}\,(mc,\, m\vec{v}\,)$$

$\mathbb{P} \cdot \mathbb{P}$ is of course an invariant

$$\mathbb{P} \cdot \mathbb{P} = m^2\,\mathbb{V} \cdot \mathbb{V} = m^2 c^2$$

Relativistic Energy-Momentum Relation

We'll see that $E = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$ and $\vec{p} = \dfrac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$

$$\mathbb{P} = \frac{(mc,\, m\vec{v}\,)}{\sqrt{1 - v^2/c^2}} = (E/c,\, \vec{p})$$

$$\mathbb{P} \cdot \mathbb{P} = (E/c)^2 - \vec{p} \cdot \vec{p}$$

but we also saw that $\mathbb{P} \cdot \mathbb{P} = m^2 c^2$

$$\Rightarrow \qquad (E/c)^2 - \vec{p} \cdot \vec{p} = m^2 c^2$$

$$E^2/c^2 - p^2 = m^2 c^2$$

$$\boxed{E^2 = p^2 c^2 + m^2 c^4}$$