Lecture 15

Identical Particles

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Two Particle States

Consider a system consisting of two particles, we'll label them 1 and 2, and two states labeled a and b.

The amplitude for particle 1 to be in state a and particle 2 to be in state b is:

$$u_a(x_1)u_b(x_2)$$

The amplitude for particle 1 to be in state b and particle 2 to be in state a is:

$$u_a(x_2)u_b(x_1)$$

Distinguishable Particles

if particle 1 and 2 are distinguishable then

particle 1 in state a and particle 2 in state b

$$u_a(x_1)u_b(x_2)$$

is distinguishable from

particle 2 in state a and particle 1 in state b

$$u_a(x_2)u_b(x_1)$$

$\Rightarrow$ square amplitudes before summing

Probability to have a particle in state a
and a particle in state b is

$$|u_a(x_1)u_b(x_2)|^2 + |u_a(x_2)u_b(x_1)|^2$$

Identical Particles

if particle 1 and 2 are indistinguishable then

particle 1 in state a and particle 2 in state b

$$u_a(x_1)u_b(x_2)$$

is indistinguishable from

particle 2 in state a and particle 1 in state b

$$u_a(x_2)u_b(x_1)$$

$\Rightarrow$ combine amplitudes before squaring

Probability to have a particle in state a
and a particle in state b is

$$|u_a(x_1)u_b(x_2) \pm u_a(x_2)u_b(x_1)|^2$$

Bosons

Bosons are particles that have integral spin

$$s = 1,\ 2,\ 3,\ \cdots$$

The wave function of a system consisting of two or more identical bosons must be symmetric under exchange of any two of the bosons.

Example of two identical bosons:

$$u(x_1, x_2) = u_a(x_1)u_b(x_2) + u_a(x_2)u_b(x_1)$$

Fermions

Fermions are particles that have half-integral spin

$$s = 1/2,\ 3/2,\ 5/2,\ \cdots$$

The wave function of a system consisting of two or more identical fermions must be antisymmetric under exchange of any two of the fermions.

Example of two identical fermions:

$$u(x_1, x_2) = u_a(x_1)u_b(x_2) - u_a(x_2)u_b(x_1)$$

Fundamental Particles

Bosons

integral spin

photon	(spin-1)
gluon	(spin-1)
W, Z	(spin-1)
Higgs	(spin-0)

Fermions

half-integral spin

electron	(spin-1/2)
quarks	(spin-1/2)
muon	(spin-1/2)
tau	(spin-1/2)
neutrinos	(spin-1/2)

Composite Particles

A composite particle made of an odd number of fermions is a fermion.

Example:

bound state of three quarks

proton, neutron, etc.

A composite particle made of an even number of fermions is a boson.

Example:

nucleus with even number of protons and neutrons

quark - antiquark bound state

pion, kaon, etc.

Summary: Probabilities for Two Particle States

Probability to have a particle in state a and a particle in state b.

Distinguishable particles:

$$|u_a(x_1)u_b(x_2)|^2 + |u_a(x_2)u_b(x_1)|^2$$

Identical Bosons:

$$|u_a(x_1)u_b(x_2) + u_a(x_2)u_b(x_1)|^2$$

Identical Fermions:

$$|u_a(x_1)u_b(x_2) - u_a(x_2)u_b(x_1)|^2$$

Probability to Have Two Particles in the Same State

Let state a and state b be the same, $a = b$.

Distinguishable particles:

$$\left|u_a(x_1)u_a(x_2)\right|^2 + \left|u_a(x_2)u_a(x_1)\right|^2 \;=\; 2\left|u_a(x_2)u_a(x_1)\right|^2$$

Identical bosons:

$$\left|u_a(x_1)u_a(x_2) + u_a(x_2)u_a(x_1)\right|^2 \;=\; \left|2\,u_a(x_1)u_a(x_2)\right|^2$$

$$=\; 4\left|u_a(x_2)u_a(x_1)\right|^2$$

Identical fermions:

$$\left|u_a(x_1)u_a(x_2) - u_a(x_2)u_a(x_1)\right|^2 \;=\; 0$$

Bosons and Fermions

Bosons:

$$\left|u_a(x_1)u_a(x_2) + u_a(x_2)u_a(x_1)\right|^2 = 4\left|u_a(x_2)u_a(x_1)\right|^2$$

The probability for two identical bosons to be in the same state is twice the probability for two distinguishable particles to be in the same state.

Fermions:

$$\left|u_a(x_1)u_a(x_2) - u_a(x_2)u_a(x_1)\right|^2 = 0$$

The number of identical fermions that can be in the same state can only be 0 or 1.

Pauli Exclusion Principle

Identical Electrons

Two electrons are identical only if they are in the same spin state with both electrons either spin up or spin down.

$\uparrow\uparrow$ or $\downarrow\downarrow$

If spins are opposite:

$\uparrow\downarrow$ or $\downarrow\uparrow$

the electrons are distinguishable and behave as distinguishable particles.

Wave Functions for Three Particle States

Distinguishable particles:

$$u_a(x_1)u_a(x_2)u_c(x_3)$$

Identical bosons:

$$u_a(x_1)u_a(x_2)u_c(x_3) + u_a(x_1)u_a(x_3)u_c(x_2)$$
$$+u_a(x_2)u_a(x_1)u_c(x_3) + u_a(x_2)u_a(x_3)u_c(x_1)$$
$$+u_a(x_3)u_a(x_1)u_c(x_2) + u_a(x_3)u_a(x_2)u_c(x_1)$$

symmetric under exchange of any two particles

Identical fermions:

$$u_a(x_1)u_a(x_2)u_c(x_3) - u_a(x_1)u_a(x_3)u_c(x_2)$$
$$-u_a(x_2)u_a(x_1)u_c(x_3) + u_a(x_2)u_a(x_3)u_c(x_1)$$
$$+u_a(x_3)u_a(x_1)u_c(x_2) - u_a(x_3)u_a(x_2)u_c(x_1)$$

antisymmetric under exchange of any two particles

Fermi Energy

The highest energy level that is occupied in the ground state of a gas of identical fermions is called the fermi energy.

In the ground state of a gas of identical bosons, all of the particles would be in the lowest energy state.

For a gas of fermions, only two particles can be in the same state (spins must be opposite). As more fermions are added higher energy states become occupied. The fermi energy will thus increase with the number of the particles in the gas.

The fermi energy is then a function of the particle density.

Fermi Energy of a One Dimensional Electron Gas

$$N \;=\; (2\ \text{spins})\,\frac{2L}{\lambda} \;=\; \frac{2L}{\pi}\,k$$

$$\mathrm{d}N \;=\; \frac{2L}{\pi}\,\mathrm{d}k \qquad \Rightarrow \qquad \mathrm{d}n \;=\; \frac{2}{\pi}\,\mathrm{d}k \;=\; \frac{2}{\pi\hbar}\,\mathrm{d}p$$

$$n = \int_0^{p_f} \frac{2}{\pi\hbar}\,\mathrm{d}p \;=\; \frac{2}{\pi\hbar}\,p_f$$

For non-relativitic gas: $p_f = \sqrt{2mE_f} \qquad \Rightarrow \qquad n \;=\; \frac{2}{\pi\hbar}\sqrt{2mE_f}$

$$\Rightarrow \qquad E_f \;=\; \frac{n^2\pi^2\hbar^2}{8m}$$

Fermi Energy of a Three Dimensional Electron Gas

$$\mathrm{d}N \;=\; 2\,\frac{L}{\pi}\,\mathrm{d}k_x\,\frac{L}{\pi}\,\mathrm{d}k_y\,\frac{L}{\pi}\,\mathrm{d}k_z \;=\; \frac{2V}{\pi^3}\,\mathrm{d}^3k \;=\; \frac{2V}{8\pi^3}\,4\pi k^2\mathrm{d}k \;=\; \frac{V}{\pi^2}\,k^2\mathrm{d}k$$

$$\mathrm{d}n \;=\; \frac{k^2}{\pi^2}\,\mathrm{d}k \;=\; \frac{p^2}{\pi^2\hbar^3}\,\mathrm{d}p$$

$$n \;=\; \int\limits_0^{p_f} \frac{p^2}{\pi^2\hbar^3}\,\mathrm{d}p \;=\; \frac{1}{3\pi^2\hbar^3}\,p_f^3$$

For non-relativitic gas: $p_f = \sqrt{2mE_f} \quad \Rightarrow \quad n \;=\; \frac{2}{\pi\hbar}\,\sqrt{2mE_f}$

$$\Rightarrow \qquad E_f \;=\; \frac{\pi^2\hbar^2}{8m}\,n^2$$

Electron Gas

$$n = \frac{p_f^3}{3\pi^2\hbar^3}$$

Non-relativisitic case: $p_f = \sqrt{2mE_f} \quad \Rightarrow \quad n = \frac{(2mE_f)^{3/2}}{3\pi^2\hbar^3}$

$$\Rightarrow \quad E_f = \frac{\hbar^2(3\pi^2)^{2/3}}{2m}\,n^{2/3}$$

Relativisitic case: $p_f = \frac{E_f}{c} \quad \Rightarrow \quad n = \frac{E_f^3}{3c^3\pi^2\hbar^3}$

$$\Rightarrow \quad E_f = \hbar c(3\pi^2)^{1/3}\,n^{1/3}$$