Lecture 11

Angular Momentum

Oct. 10, 2007

Review: The Schrodinger Equation for the Hydrogen Atom

Schrodinger equation for the hydrogen atom

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right) u(r,\theta,\phi) = E\, u(r,\theta,\phi)$$

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right] + V(r)\right\} u(r,\theta,\phi)$$

$$= E\, u(r,\theta,\phi)$$

with $$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

Solve by separation of variables

$$u(r,\theta,\phi) = R(r)Y(\theta,\phi) = R(r)F(\theta)\Phi(\phi)$$

Review: The Differential Equations for $R(r),\ F(\theta),\ \Phi(\phi)$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{2m}{\hbar^2}\left[V(r) - E\right] + \frac{\lambda}{r^2}\right) R(r) = 0$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \cot\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta} - \lambda - \frac{m^2}{\sin^2\theta}\right) F(\theta) = 0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\,\Phi(\phi) = -m^2\Phi(\phi)$$

Review: Solution for $\Phi(\phi)$

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\,\Phi(\phi) \;=\; -m^2\Phi(\phi)$$

Solution:

$$\Phi_m(\phi) = Ce^{im\phi}$$

A quantization condition:

$\Phi(\phi)$ will only be single-valued if m is an integer

$$m = 0,\ \pm1,\ \pm2,\ \pm3,\ \cdots$$

Review: Solution for $F(\theta)$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \cot\theta\,\frac{\mathrm{d}}{\mathrm{d}\theta} - \lambda - \frac{m^2}{\sin^2\theta}\right) F(\theta) \;=\; 0$$

Another quantization condition:

This will have well-behaved (finite) solutions only if :

$$\lambda = -l(l+1) \qquad\qquad l = 0,\ 1,\ 2,\ 3, \cdots$$

and

$$|m| \le l \qquad\qquad m = -l, \cdots,\ -1,\ 0,\ 1,\ \cdots,\ l$$

Review: Spherical Harmonics

The spherical harmonics:

$$Y_{lm}(\theta,\phi) = F_l^m(\theta)\Phi_m(\phi) = F_l^m(\theta)e^{im\phi}$$

are the angular part of the energy eigenfunction solutions for any case of a spherically symmetric potential (V depending only on r).

The probability of the particle at the angular location θ, ϕ is given by the square of the spherical harmonic.

$$P_{lm}(\theta,\phi) = |Y_{lm}(\theta,\phi)|^2$$

Note that it depends on the particular values of l and m.

Eigenfunctions

$$Y_{lm}(\theta,\phi) = F_l^m(\theta)e^{im\phi}$$

$$\Rightarrow \qquad \frac{\partial}{\partial\phi} Y_{lm}(\theta,\phi) = \frac{\partial}{\partial\phi} F_l^m(\theta)e^{im\phi} = im\, F_l^m(\theta)e^{im\phi} = im\, Y_{lm}(\theta,\phi)$$

$\Rightarrow \quad Y_{lm}(\theta,\phi)$ is an eigenfunction

of the operator $\dfrac{\partial}{\partial\phi}$ with eigenvalue im

Eigenfunctions

$$\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} - \lambda - \frac{m^2}{\sin^2\theta}\right) F_l^m(\theta) \;=\; 0$$

$$\left(\frac{\partial^2}{\mathrm{d}\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} - \lambda - \frac{m^2}{\sin^2\theta}\right) F_l^m(\theta)\Phi_m(\phi) \;=\; 0$$

$$\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} - \lambda + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{lm}(\theta,\phi) \;=\; 0$$

$$\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{lm}(\theta,\phi) \;=\; \lambda\, Y_{lm}(\theta,\phi)$$

$\Rightarrow \quad Y_{lm}(\theta,\phi)$ is an eigenfunction

of the operator $\quad \dfrac{\partial^2}{\partial\theta^2} + \cot\theta\,\dfrac{\partial}{\partial\theta} + \dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2} \quad$ with eigenvalue $\quad \lambda = -l(l+1)$

Angular Momentum Operators

Classically: $\vec{L} = \vec{r} \times \vec{p}$

So for the quantum mechanical operator

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

$$\hat{L}_x = -i\hbar\left(y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y}\right) \qquad \hat{L}_y = -i\hbar\left(z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z}\right) \qquad \hat{L}_z = -i\hbar\left(x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\right)$$

The L_z Operator

$$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial\phi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial\phi}$$

$$= \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}x + \frac{\partial}{\partial z}(0) = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}}$$

$$\hat{L}_z Y_{lm}(\theta,\phi) = m\hbar\, Y_{lm}(\theta,\phi)$$

$Y_{lm}(\theta,\phi)$ is an eigenfunction of the L_z operator with eigenvalue $m\hbar$.

The L^2 Operator

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

It can be shown that

$$\hat{L}^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \, \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$\hat{L}^2 \, Y_{lm}(\theta, \phi) = l(l+1) \, \hbar^2 \, Y_{lm}(\theta, \phi)$$

$Y_{lm}(\theta, \phi)$ is an eigenfunction of the L^2 operator with eigenvalue $l(l+1)\hbar^2$.

Angular Momentum Eigenstates

$Y_{lm}(\theta, \phi)$ represents a state of:

Definite magnitude of angular momentum: $|\vec{L}| = \sqrt{l(l+1)}\,\hbar$

Definite z-component of angular momentum: $L_z = m\,\hbar$

For Example, $l = 1$ $m = 1$

$Y_{11}(\theta, \phi)$ is a state with

$$|\vec{L}| = \sqrt{2}\,\hbar \qquad L_z = \hbar$$

Example of l=1

- A particle can be prepared in a state with definite magnitude of angular momentum and definite value of angular momentum in some direction (conventionally chosen to be the z-axis).

- If a particle has a definite angular momentum along some direction, it can not have a definite angular momentum along any other direction.

- As a result, the angular momentum cannot point in a definite direction. If it did, it would have a definite value along that direction, say z, and a definite definite value along x and y also, namely, zero.

- When the angular momentum is measured along any direction, the only possible values that will be measured are the various possible values of $m\hbar$. If the particle is in a state of definite angular momentum along a particular direction then that value of $m\hbar$ is certain to be measured in that direction.

- If the particle is not in a state of definite angular momentum along that direction then the various values of $m\hbar$ will be measured with relative probabilities dependent on the state.

The Strange Case of QM Angular Momentum

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