Lecture 9

Heisenberg Uncertainty Principle

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Heisenberg Uncertainty Principle

For any state:

$$\boxed{\Delta x \, \Delta p \sim \hbar}$$

This is a fundamental limitation in quantum mechanics.

The uncertainty in the position of a particle times its uncertainty in monentum cannot be determined to better than about $\hbar$.

It follows from the fact that, in quantum mechanics, particles have a wave-like nature and that for waves of any type.

$$\Delta x \, \Delta k \sim 1$$

So far, we've been a bit vague. We haven't defined exactly what Δx and Δp are.

Let's be more quantitative about this.

Standard Deviation

The standard deviation, σ, is a convenient measure of the width of a distribution.

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Examples:

Flat distribution of width L:

$$\sigma^2 = \frac{\int_0^L x^2 \,\mathrm{d}x}{\int_0^L \mathrm{d}x} - \left(\frac{\int_0^L x \,\mathrm{d}x}{\int_0^L \mathrm{d}x} \right)^2 = \frac{L^2}{3} - \frac{L^2}{2} = \frac{L^2}{12} \qquad \Rightarrow \qquad \sigma = \frac{L}{\sqrt{12}}$$

Distribution consisting of equal amounts of p and $-p$:

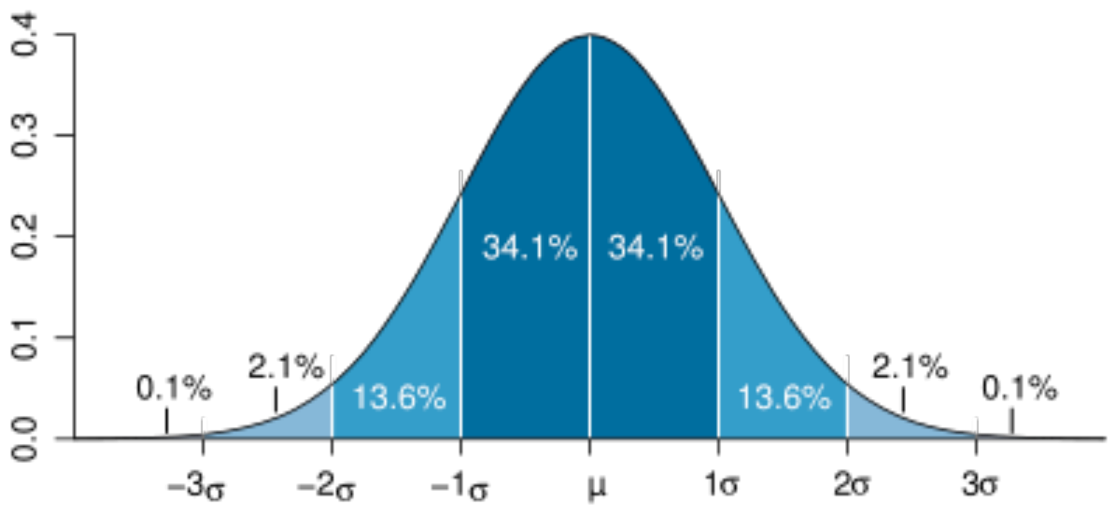
$$\langle p \rangle = 0 \qquad \langle p^2 \rangle = p^2 + (-p)^2 = 2\,p^2$$

$$\sigma = \sqrt{2}\,p$$

Gaussian

Gaussian function:

$$Ae^{-(x-\mu)^2/2\sigma^2}$$



Uncertainty Relation for Gaussian Distributions

If momentum distribution of a particle, $|A(p)|^2$, is Gaussian then the position distribution of the particle, $|u(x)|^2$, is Gaussian and vice versa.

Take $\quad |A(p)|^2 \sim e^{-p^2/2\sigma_p^2} \quad \Rightarrow \quad A(p) \sim e^{-p^2/4\sigma_p^2}$

$$u(x) \sim \int_{-\infty}^{\infty} A(p)\, e^{ipx/\hbar}\, \mathrm{d}p = \int_{-\infty}^{\infty} e^{-p^2/4\sigma_p^2}\, e^{ipx/\hbar}\, \mathrm{d}p \sim e^{-x^2\sigma_p^2/\hbar^2}$$

$$\Rightarrow \quad |u(x)|^2 \sim e^{-x^2 2\sigma_p^2/\hbar^2} = e^{-x^2/2\sigma_x^2} \quad \Rightarrow \quad \sigma_x^2 = \frac{\hbar^2}{4\,\sigma_p^2}$$

$$\sigma_x\,\sigma_p = \frac{\hbar}{2} \quad \text{for Gaussian distributions}$$

Summary of the Heisenberg Uncertainty Relation

- Define the uncertainty of the position of a particle or the momentum of a particle to be the standard deviation of the corresponding probability distribution.

- If the position distribution is Gaussian then so is the position distribution and vice versa.

- For Gaussian distributions the product of the momentum and position uncertainties is minimum.

$$\sigma_x \, \sigma_p = \frac{\hbar}{2}$$

- For any other distributions the uncertainty product will be larger

$$\sigma_x \, \sigma_p > \frac{\hbar}{2}$$

- In general,

$$\boxed{\sigma_x \, \sigma_p \geq \frac{\hbar}{2}}$$

Energy Time Uncertainty Relation

Since energy is to time as momentum is to position, there is also a lower bound on the product of the energy and time uncertainty.

$$\Delta E\,\Delta t \;\geq\; \frac{\hbar}{2}$$

- A particle in a state of definite energy has equal probability of existing at all times.
- If a state has a finite lifetime then it cannot therefore have a definite energy.
- This means for example that the spectral lines emitted in the decay of excited atoms do not have definite frequency. There must be a spread in frequency such that

$$h\,\Delta f \geq \frac{\hbar}{2\tau}$$ where τ is the lifetime of the excited state.

$$\Delta f \geq \frac{1}{4\pi\tau}$$

Mass Width of Unstable Particles

The energy-time uncertainty relation provide a means of measuring the lifetime of very short lived particles.
If a particle lives for a lifetime τ then its energy and therefore its mass is uncertain by

$$\Delta mc^2\,\tau \sim \hbar \qquad \Rightarrow \qquad \Delta m \sim \frac{\hbar}{\tau c^2}$$

For example, the Z-boson can be produced in collisions of electrons and positrons (anti-electrons). It mass width can be determined by measuring the width of the Z-resonance. In the case of the Z, the width of the resonance Δmc^2 is 2.5 GeV.

$$\Rightarrow \qquad \tau \sim \frac{\hbar}{\Delta mc^2} = \frac{\hbar c}{\Delta mc^2 c} = \frac{(0.2 \times 10^{-15}\ \text{GeV}\cdot\text{m})}{(2.5\ \text{GeV})(3.0 \times 10^{8}\ \text{m/s})} = 3{\times}10^{-25}\ \text{s}$$

There is no other way to measure such a small lifetime.