Lecture 8

Wave Packets

Oct. 1, 2007

Review: Wave Function

State specified by:

$$\psi(x,t)$$

Probability:

$$P(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\psi(x,t)$$

Normalization:

$$\int\limits_{-\infty}^{\infty} P(x,t)\,\mathrm{d}x = \int\limits_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\,\mathrm{d}x = 1$$

Review: Schrodinger Equation

Operators:

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \qquad\qquad \hat{p} = -i\hbar \frac{\partial}{\partial x}$$

Schrodinger Equation:

$$\hat{E}\,\psi(x,t) = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\psi(x,t)$$

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)$$

Energy Eigenfunction Solution (state of definite energy):

$$\psi(x,t) = u_E(x)e^{-iEt/\hbar}$$

$$\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)u_E(x) = E\,u_E(x)$$

Review: Free Particle

$$\boxed{V(x) = 0 \qquad \text{for all } x}$$

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_E(x) = E\,u_E(x)$$

Solutions:

$$u_E(x) = Ae^{ipx/\hbar} \qquad \longrightarrow \qquad \psi(x,t) = Ae^{i(px-Et)/\hbar}$$

or

$$u_E(x) = Ae^{-ipx/\hbar} \qquad \longrightarrow \qquad \psi(x,t) = Ae^{i(-px-Et)/\hbar}$$

$$E = \frac{p^2}{2m}$$

.

Review: Particle in Infinite Well

$V(x) = 0$ for $0 < x < L$

$V(x) = \infty$ for $x < 0$ or $x > L$

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_E(x) = E\,u_E(x) \qquad \text{for } 0 < x < L$$

Boundary Conditions:

$$u_E(0) = 0 \qquad u_E(L) = 0$$

Solution:

$$u_n(x) = \frac{A}{2i}\left(e^{ip_n x/\hbar} - e^{-ip_n x/\hbar}\right) = A\sin\left(\frac{p_n x}{\hbar}\right) = A\sin\left(k_n x\right)$$

$$k_n = \frac{n\pi}{L} \qquad p_n = \frac{n\pi\hbar}{L} \qquad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$\psi_n(x,t) = A\sin\left(\frac{n\pi x}{L}\right)e^{-iE_n t/\hbar} \qquad A = \sqrt{\frac{2}{L}}$$

Expectation Values

- Any observable (e.g., energy, momentum, position, angular momentum) is represented by a corresponding operator.

- For a particle in a given state, in general, we can't make exact predictions for what value of an observable will be measured. We can only give the probability distribution for the observable.

- The expectation value of an observable is the average value of the observable that will be observed for a particle in a given state.

$$\boxed{\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{A}\,\psi(x,t)\,\mathrm{d}x}$$

Examples of Expectation Values

$$\langle E\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{E}\,\psi(x,t)\,\mathrm{d}x \;=\; i\hbar \int_{-\infty}^{\infty} \psi^*(x,t)\,\frac{\partial}{\partial t}\,\psi(x,t)\,\mathrm{d}x$$

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{p}\,\psi(x,t)\,\mathrm{d}x \;=\; -\,i\hbar \int_{-\infty}^{\infty} \psi^*(x,t)\,\frac{\partial}{\partial x}\,\psi(x,t)\,\mathrm{d}x$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{p}^2\,\psi(x,t)\,\mathrm{d}x \;=\; -\,\hbar^2 \int_{-\infty}^{\infty} \psi^*(x,t)\,\frac{\partial^2}{\partial x^2}\,\psi(x,t)\,\mathrm{d}x$$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{x}\,\psi(x,t)\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} \psi^*(x,t)\,x\,\psi(x,t)\,\mathrm{d}x$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\,\hat{x}^2\,\psi(x,t)\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} \psi^*(x,t)\,x^2\,\psi(x,t)\,\mathrm{d}x$$

Localized States

- As we've seen, a free particle having a definite momentum

  $$u(x) = e^{ipx/\hbar}$$

  is not localized. It has equal probability to be anywhere!

- How can we make a free particle state that is localized?

- Just as you can make a localized wave pulse (e.g., wave pulse on a string), by combining waves of different wave numbers (wavelengths), you can make a localized particle state by combining states of different momentum.

  $$u(x) = A_1 e^{ip_1 x/\hbar} + A_2 e^{ip_2 x/\hbar} + \cdots$$

- Note that this localized state is not then a state of definite momentum.

Wave Packet

- Now go to the case where the sum is over a continuous range of momentum.

- The discreet sum over momentum now becomes an integral over momentum.

$$\boxed{u(x) = \int_{-\infty}^{\infty} A(p)\, e^{ipx/\hbar}\, \mathrm{d}p}$$

where $A(p)$ is the weighting factor for each momentum.

- It can be shown that the probability for a particle in this state to have momentum p is given by

$$P(p) = |A(p)|^2$$

Wave Packet with a Flat Momentum Distribution

Take the case of:

$$A(p) = C \qquad \text{for} \quad -\frac{\Delta p}{2} < p < \frac{\Delta p}{2}$$

$$A(p) = 0 \qquad \text{elsewhere}$$

$$u(x) = \int_{-\infty}^{\infty} A(p)\, e^{ipx/\hbar}\, \mathrm{d}p \;=\; \int_{-\Delta p/2}^{\Delta p/2} C\, e^{ipx/\hbar}\, \mathrm{d}p$$

$$= C\,\frac{\hbar}{ix}\left(e^{i\Delta px/2\hbar} - e^{-i\Delta px/2\hbar}\right) \;=\; \frac{2\,C\hbar}{x}\,\sin\left(\frac{\Delta px}{2\hbar}\right)$$

$$|u(x)|^2 = \frac{4\,C^2\hbar^2}{x^2}\,\sin^2\left(\frac{\Delta px}{2\hbar}\right)$$

Uncertainty Relation

The distribution:

$$|u(x)|^2 = \frac{4\,C^2\hbar^2}{x^2}\,\sin^2\left(\frac{\Delta p x}{2\hbar}\right)$$

goes to zero at: $x = \frac{\pm 2\pi\hbar}{\Delta p} \qquad \Rightarrow \qquad \Delta x \sim \frac{4\pi\hbar}{\Delta p}$

$$\Delta x \Delta p \sim 4\pi\hbar$$

- Δx and Δp are inversely related
- Larger Δp $\rightarrow$ smaller Δx and vice versa
- The better we know Δp the less we know Δx and vice versa.
- If we know p exactly, $\Delta p = 0$, we don't know x at all, $\Delta x = \infty$, and vice versa.

Uncertainty Relation for Infinite Well

Consider a particle in an infinite well of width L.

The particle is localized within the well so $\Delta x \sim L$.

Doesn't the particle have a definite momentum, $p = \dfrac{\pi\hbar}{L}$,

and therefore isn't $\Delta p = 0 \quad \Rightarrow \quad \Delta x \, \Delta p = 0$

No!

$$u(x) \sim \sin\left(\frac{px}{\hbar}\right) \sim \left(e^{ipx/\hbar} - e^{-ipx/\hbar}\right)$$

The state is a mixture of two momentum state, $p = \pm\pi\hbar/L$

There is a 50% probability for the particle to have $p = +\pi\hbar/L$

and a 50% probability for the particle to have $p = -\pi\hbar/L$

$$\Delta p \sim p = \frac{\pi\hbar}{L} \qquad \Rightarrow \qquad \Delta x \, \Delta p \sim \pi\hbar$$

Heisenberg Uncertainty Principle

For any state:

$$\boxed{\Delta x \, \Delta p \sim \hbar}$$

This is a fundamental limitation in quantum mechanics.

The uncertainty in the position of a particle times its uncertainty in monentum cannot be determined to better than about $\hbar$.

It follows from the fact that, in quantum mechanics, particles have a wave-like nature and that for waves of any type.

$$\Delta x \, \Delta k \sim 1$$

So far, we've been a bit vague. We haven't defined exactly what Δx and Δp are.

We'll be more quantitative about this next lecture.