

Homework Assignment #9
Physics 273

- 3.10) Find the kinetic energy of a particle with a mass of one gram moving with half the speed of light. Compare your answer with one you would find using the nonrelativistic formula.

$$E = \gamma mc^2 \qquad K = E - mc^2 = (\gamma - 1)mc^2$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (1/2)^2}} = 1.155$$

$$K = (1.155 - 1)(10^{-3} \text{ kg})(3.0 \times 10^8 \text{ m/s})^2 = 1.39 \times 10^{13} \text{ J}$$

Nonrelativistically:

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m(c/2)^2 = \frac{1}{8} (10^{-3} \text{ kg})(3.0 \times 10^8 \text{ m/s})^2 = 1.13 \times 10^{13} \text{ J}$$

3.13) What is the total energy of a particle with a rest mass of 1 gram moving with half the speed of light?

$$E = \gamma mc^2$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (1/2)^2}} = 1.155$$

$$K = (1.155)(10^{-3} \text{ kg})(3.0 \times 10^8 \text{ m/s})^2 = 1.04 \times 10^{14} \text{ J}$$

- 3.18) Suppose that we could use the energy released when 1 g of antimatter annihilates 1 g of matter to lift a mass from the Earth's surface. How much mass could be lifted?

$$\text{Energy released} = 2mc^2 = (2)(10^{-3} \text{ kg})(3.0 \times 10^8)^2 = 1.8 \times 10^{14} \text{ J}$$

$$mgh = 1.8 \times 10^{14} \text{ J} \quad \Rightarrow \quad m = \frac{(1.8 \times 10^{14} \text{ J})}{(9.8 \text{ m/s}^2)(10^3 \text{ m})} = 1.8 \times 10^{10} \text{ kg}$$

- 3.23) Find the minimum energy a gamma ray must have to initiate the reaction $\gamma + p \rightarrow \pi^0 + p$ if the target proton is at rest. Note that $m_p c^2 = 940$ MeV and $m_\pi c^2 = 140$ MeV.

For the minimum energy, in the center of mass frame the π^0 and the p will be at rest.

$$\Rightarrow s = (m_\pi c^2 + m_p c^2)^2 = (m_\pi + m_p)^2 c^4$$

$$s = (\mathbb{P}_\gamma + \mathbb{P}_p)^2 = \mathbb{P}_\gamma^2 + \mathbb{P}_p^2 + 2\mathbb{P}_\gamma \cdot \mathbb{P}_p = (m_p c^2)^2 + 2E_\gamma m_p c^2$$

$$\Rightarrow (m_p c^2)^2 + 2E_\gamma m_p c^2 = (m_\pi + m_p)^2 c^4$$

$$\Rightarrow E_\gamma = \frac{(m_\pi^2 + 2m_p m_\pi) c^2}{2m_p} = \frac{m_\pi (m_\pi + 2m_p) c^2}{2m_p}$$

$$= \frac{(140 \text{ MeV})(140 \text{ MeV} + (2)(940 \text{ MeV}))}{(2)(940 \text{ MeV})} = 150 \text{ MeV}$$

- 3.32) Suppose that a π^0 decays into two massless photons. If one of the photons moves in the same direction that the π^0 was moving in, what is the direction of motion of the second one? What are the energies of the photons in terms of the momentum of the original π^0 ?

By conservation of momentum, if one of the photons is moving in the same direction as the mother π^0 , it can't have a component of momentum perpendicular to the π^0 since then momentum in that direction wouldn't be conserved. So the second photon must be moving in either the same or opposite direction of the first one.

$$(\mathbb{P}_1 + \mathbb{P}_2)^2 = m_\pi^2 c^4 \quad \Rightarrow \quad 2\mathbb{P}_1 \cdot \mathbb{P}_2 = 2E_1 E_2 (1 - \cos \theta) = m_\pi^2 c^4$$

$$\cos \theta \text{ cannot be } 1 \quad \Rightarrow \quad \theta = \pi \quad \Rightarrow \quad 4E_1 E_2 = m_\pi^2 c^4$$

$$p_1 + p_2 = p_\pi \quad \Rightarrow \quad E_1 - E_2 = p_\pi c \quad \Rightarrow \quad E_{2,1} = E_{1,2} - p_\pi c$$

$$\Rightarrow \quad E_{1,2}(E_{1,2} - p_\pi c) = \frac{m_\pi^2 c^4}{4} \quad \Rightarrow \quad E_{1,2}^2 - E_{1,2} p_\pi c - \frac{m_\pi^2 c^4}{4} = 0$$

$$\text{Use quadratic formula} \quad E_{1,2} = \frac{p_\pi c \pm \sqrt{p_\pi^2 c^2 - m_\pi^2 c^4}}{2}$$

$$\Rightarrow \quad E_1 = \frac{p_\pi c + \sqrt{p_\pi^2 c^2 - m_\pi^2 c^4}}{2} \quad E_2 = \frac{p_\pi c - \sqrt{p_\pi^2 c^2 - m_\pi^2 c^4}}{2}$$

$$\text{Check} \quad E_1 E_2 = \frac{(p_\pi c + \sqrt{p_\pi^2 c^2 - m_\pi^2 c^4})(p_\pi c - \sqrt{p_\pi^2 c^2 - m_\pi^2 c^4})}{4} = \frac{m_\pi^2 c^4}{4}$$

- 3.33) A particle whose rest energy is $Mc^2 = 765$ MeV decays into a π^0 with a rest energy of 135 MeV, and another particle a π^+ with a rest energy of 140 MeV. What will be the momenta of the decay products?

By conservation of momentum $p_{\pi^0} = -p_{\pi^+} = p$

By conservation of energy $E_{\pi^0} + E_{\pi^+} = Mc^2$

$$\Rightarrow \sqrt{m_{\pi^0}^2 c^4 + p^2 c^2} + \sqrt{m_{\pi^+}^2 c^4 + p^2 c^2} = Mc^2$$

This is a bit hard to solve. Easier if we use the result of problem 3.34.

$$E_{\pi^0} = \frac{(M^2 + m_{\pi^0}^2 - m_{\pi^+}^2)c^2}{2M}$$

$$= \frac{[(765 \text{ MeV}/c^2)^2 + (135 \text{ MeV}/c^2)^2 - (140 \text{ MeV}/c^2)^2] c^2}{(2)(765 \text{ MeV}/c^2)} = 382 \text{ MeV}$$

$$pc = \sqrt{E_{\pi^0}^2 - m_{\pi^0}^2 c^4} = \sqrt{(382 \text{ MeV})^2 - (135 \text{ MeV})^2} = 357 \text{ MeV}$$

- 3.34) A particle of mass M decays at rest into two particles of mass m_1 and m_2 . Show that energy and momentum conservation imply that the energy of particle 1 is given by:

$$E_1 = \frac{(M^2 + m_1^2 - m_2^2)c^2}{2M}$$

By conservation of energy $E_1 + E_2 = Mc^2 \Rightarrow E_2 = Mc^2 - E_1$

$$\Rightarrow E_2^2 = M^2c^4 - 2Mc^2E_1 + E_1^2$$

$$E_1^2 = m_1^2c^4 + p_1^2c^2 \qquad E_2^2 = m_2^2c^4 + p_2^2c^2$$

$$\Rightarrow m_2^2c^4 + p_2^2c^2 = M^2c^4 - 2Mc^2E_1 + m_1^2c^4 + p_1^2c^2$$

By conservation of momentum $p_1 + p_2 = 0 \Rightarrow p_1 = -p_2 = p$

$$\Rightarrow m_2^2c^4 = M^2c^4 - 2Mc^2E_1 + m_1^2c^4$$

$$\Rightarrow 2ME_1 = M^2c^2 + m_1^2c^2 - m_2^2c^2$$

$$\Rightarrow E_1 = \frac{(M^2 + m_1^2 - m_2^2)c^2}{2M}$$

- 3.35) In a system of two particles with energies and momenta (E_1, \vec{p}_1) and (E_2, \vec{p}_2) , respectively, the quantity

$$s = (E_1 + E_2)^2 - c^2|\vec{p}_1 + \vec{p}_2|^2$$

is invariant, that is, has the same numerical value in all reference frames.

- a) Consider a center-of-mass collision of a proton and an antiproton ($mc^2 = 938.3$ MeV). What is the minimum momentum required to produce a particle with mass $Mc^2 = 91.2$ GeV?

In center-of-mass frame $\vec{p}_1 = -\vec{p}_2$ and since $m_1 = m_2$ $E_1 = E_2 = E$

$$s = M^2c^4 = (E_1 + E_2)^2 = 4E^2 \quad \Rightarrow \quad E = \frac{Mc^2}{2} = 45.6 \text{ GeV}$$

- b) In a fixed-target accelerator, an antiproton projectile collides with a proton target at rest. What is the minimum energy that the antiproton must have to create the new particle of part (a)?

In lab frame $E_1 = E$ $E_2 = m_p c^2$ $p_1 = \sqrt{E^2 - m_p^2 c^4}/c = p$ $p_2 = 0$

$$\Rightarrow s = M^2c^4 = (E + m_p c^2)^2 - c^2 p^2 = E^2 + 2m_p E + m_p^2 c^4 - c^2 p^2$$

$$= 2m_p c^2 E \quad \Rightarrow \quad E = \frac{M^2 c^4}{2m_p c^2} = \frac{(91.2 \text{ GeV})^2}{(2)(0.94 \text{ GeV})} = 4400 \text{ GeV}$$