Homework Assignment #7
Physics 273

10.7) Consider the case of two identical fermions, both in the spin-up state in a one-diemsnional infinitely deep well of width, $2a$. Write the wave function for the lowest energy state. For what value(s) of position does the wave function vanish?

Since these are identical fermions the overall wave function must be anti-symmetric. The spin part of the wave function is symmetric because the spins are aligned, therefore, the space part of the wave function must be antisymmetric.

$$u(x_1, x_2) = \frac{1}{\sqrt{2}}\left[u_1(x_1)u_2(x_2) - u_1(x_2)u_2(x_1)\right]$$

where $u_1(x)$ and $u_2(x_2)$ are the ground state and

the first excited state wave functions, respectively.

the factor of $1/\sqrt{2}$ is there for normalization.

$$u(x_1, x_2) = \frac{1}{\sqrt{2}}\frac{2}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right]$$

$$= \frac{1}{\sqrt{2}a}\left[\sin\left(\frac{\pi x_1}{2a}\right)\sin\left(\frac{\pi x_2}{a}\right) - \sin\left(\frac{\pi x_2}{2a}\right)\sin\left(\frac{\pi x_1}{a}\right)\right]$$

The zero's are at $\quad x = 0,\ a,\ 2a$

10.12) Calculate the Fermi energy for potassium, for which the free-electron density is 1.40×10^{28} electrons/m^3.

$$E_f = \frac{\hbar^2}{2m}(3\pi^2 n_f)^{2/3} = \frac{\hbar^2 c^2}{2mc^2}(3\pi^2 n_f)^{2/3}$$

$$= \frac{(2.0 \times 10^{-7} \text{ eV}\cdot\text{m})^2}{(2)(5.1 \times 10^5 \text{ eV})}[9\pi^4(14.0 \times 10^{27} \text{ m}^{-3})^2]^{1/3} = 2.2 \text{ eV}$$

10.18) Use dimensional analysis to show that the Fermi energy for nonrelativistic particles must be proportional to $n^{2/3}$. (*Hint*: The energy can only involve $\hbar$, m and the interparticle spacing a and must therefore be of the form $E = \hbar^\alpha m^\beta a^\gamma$.)

$$E\left[m \cdot l^2 \cdot t^{-2}\right] \qquad \hbar\left[E \cdot t\right] \;\Rightarrow\; \hbar\left[m \cdot l^2 \cdot t^{-1}\right] \;\Rightarrow\; \frac{\hbar}{ma^2}\left[t^{-1}\right]$$

$$\Rightarrow\; E \propto ma^2\left(\frac{\hbar}{ma^2}\right)^2 = \frac{\hbar^2}{ma^2} \;\Rightarrow\; E \propto a^{-2} \propto \left[\left(\frac{1}{n}\right)^{1/3}\right]^{-2} = n^{2/3}$$

10.20) Consider a gas of free massless particles (as in Example 10-6). Use dimensional analysis to show that the Fermi energy must be proportional to $n^{1/3}$. (*Hint*: The energy can only involve $\hbar$, m and the interparticle spacing a.)

$$\hbar\,[E \cdot t] \;\Rightarrow\; \hbar c\,[E \cdot l] \;\Rightarrow\; \frac{\hbar c}{a}\,[E]$$

$$\Rightarrow\; E \propto \frac{\hbar c}{a} \propto \left[\left(\frac{1}{n}\right)^{1/3}\right]^{-1} = n^{1/3}$$

10.24) Calculate the radius of an electron-degenerate star of mass 0.8×10^{30} kg.

$$R = \left(\frac{81\pi^2}{16}\right)^{1/3} \frac{\hbar^2}{G m_p^2 m_e} N^{-1/3} \approx (1.15 \times 10^{23}\ \text{km})\, N^{-1/3}$$

number of nucleons in the star: $N = \dfrac{m_{\text{star}}}{m_p} = \dfrac{0.8 \times 10^{30}\ \text{kg}}{1.7 \times 10^{-27}\ \text{kg}} = 0.47 \times 10^{57}$

$$R \approx (1.15 \times 10^{23}\ \text{km})\,(0.47 \times 10^{57})^{-1/3} = 1.5 \times 10^{4}\ \text{km}$$

10.25) Calculate the radius of a neutron star. The calculation is essentially the same as that for a electron-degenerate star, with the following changes: (i) Replace m_e by m_n; (ii) replace the number of electrons N_e, which was taken to be $N/2$ (where N is the number of protons plus the number of neutrons in the star), by N. Calculate the numerical value of the radius for a star of mass 4×10^{30} kg.

$$R = \left(\frac{81\pi^2}{16}\right)^{1/3} \frac{\hbar^2}{G m_p^2 m_n} (2N)^{-1/3} \approx (2)^{-1/3} \frac{m_e}{m_n} \left(1.15 \times 10^{23}\ \text{km}\right) N^{-1/3}$$

$$= (0.79) \frac{(9.11 \times 10^{-31}\ \text{kg})}{(1.67 \times 10^{-27}\ \text{kg})} \left(1.15 \times 10^{23}\ \text{km}\right) N^{-1/3} = \left(5.0 \times 10^{19}\ \text{km}\right) N^{-1/3}$$

$$\text{number of nucleons in the star:} \quad N = \frac{m_{\text{star}}}{m_p} = \frac{4.0 \times 10^{30}\ \text{kg}}{1.7 \times 10^{-27}\ \text{kg}} = 2.4 \times 10^{57}$$

$$R \approx \left(5 \times 10^{19}\ \text{km}\right) \left(2.4 \times 10^{57}\right)^{-1/3} = 6.7\ \text{km}$$

11.2) Consider the ground state of the silicon atom (Z=14). What is the electron configuration of this state?

$$1s^2 2s^2 2p^6 3s^2 3p^2$$

11.5) X-rays bombarding heavy atoms can be used to eject electrons from the $1s$ shell in the atoms. Estimate the maximum wavelength of photons required to eject an electron from the $1s$ shell of copper, for which $Z = 29$.

The inner most electron see the unshielded nucleus so the binding energy is approximately

$$E = -\frac{1}{2}m_e c^2 (Z\alpha)^2 = -(13.6\ \text{eV})(29)^2 = -11.4\ \text{keV}$$