Homework Assignment #4
Physics 273

6.2 Consider the Gaussian wave function $u(x) = Ce^{-x^2/2a^2}$

a) Calculate $\langle x \rangle$.

$$\langle x \rangle = \int_{-\infty}^{\infty} u^*(x)\,\hat{x}\,u(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} u^*(x)\,x\,u(x)\,\mathrm{d}x = C^2 \int_{-\infty}^{\infty} x\,e^{-x^2/a^2}\,\mathrm{d}x$$

$$= C^2 \int_{0}^{\infty} x\,e^{-x^2/a^2}\,\mathrm{d}x + C^2 \int_{-\infty}^{0} x\,e^{-x^2/a^2}\,\mathrm{d}x$$

$$= C^2 \int_{0}^{\infty} x\,e^{-x^2/a^2}\,\mathrm{d}x - C^2 \int_{0}^{\infty} x\,e^{-x^2/a^2}\,\mathrm{d}x = 0$$

could have immediately gotten $\langle x \rangle = 0$ from the fact
that $u(x) = Ce^{-x^2/2a^2}$ is symmetric about $x = 0$.

b) Calculate $\langle x^2 \rangle$.

$$\langle x^2 \rangle = \int\limits_{-\infty}^{\infty} u^*(x)\, \hat{x}^2\, u(x)\, \mathrm{d}x = \int\limits_{-\infty}^{\infty} u^*(x)\, x^2\, u(x)\, \mathrm{d}x = C^2 \int\limits_{-\infty}^{\infty} x^2\, e^{-x^2/a^2}\, \mathrm{d}x = \frac{C^2 a^3 \sqrt{\pi}}{2}$$

where I used $\displaystyle\int\limits_{-\infty}^{\infty} x^2\, e^{-Ax^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2A^{3/2}}$ from Appendix B, page 577

to find C^2 normalize $u(x)$ $\Rightarrow$ $\displaystyle\int\limits_{-\infty}^{\infty} u^*(x)\, u(x)\, \mathrm{d}x = C^2 \int\limits_{-\infty}^{\infty} e^{-x^2/a^2}\, \mathrm{d}x = 1$

using $\displaystyle\int\limits_{-\infty}^{\infty} e^{-Ax^2}\, \mathrm{d}x = \sqrt{\frac{\pi}{A}}$ also from page 577 $\Rightarrow$ $\displaystyle C^2 = \frac{1}{\sqrt{\pi} a}$

$$\Rightarrow \quad \langle x^2 \rangle = \frac{a^2}{2}$$

c) Calculate $\langle p \rangle$.

$$\langle p \rangle = \int\limits_{-\infty}^{\infty} u^*(x)\, \hat{p}\, u(x)\, \mathrm{d}x = -i\hbar \int\limits_{-\infty}^{\infty} u^*(x)\, \frac{\mathrm{d}}{\mathrm{d}x}\, u(x)\, \mathrm{d}x$$

$$= -i\hbar\, C^2 \int\limits_{-\infty}^{\infty} e^{-x^2/2a^2}\, \frac{\mathrm{d}}{\mathrm{d}x}\, e^{-x^2/2a^2}\, \mathrm{d}x$$

$$= -i\hbar\, C^2 \int\limits_{-\infty}^{\infty} \left(\frac{-x}{a^2}\right) e^{-x^2/a^2}\, \mathrm{d}x = \frac{\hbar}{\sqrt{\pi} a^3} \int\limits_{-\infty}^{\infty} x\, e^{-x^2/a^2}\, \mathrm{d}x = 0$$

$$\text{since from part a)} \qquad \int\limits_{-\infty}^{\infty} x\, e^{-x^2/a^2}\, \mathrm{d}x = 0.$$

also from symmetry no preference between $+p$ and $-p$ $\quad \Rightarrow \quad \langle p \rangle = 0$

d) Calculate $\langle p^2 \rangle$.

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} u^*(x)\, \hat{p^2}\, u(x)\, \mathrm{d}x = -\hbar^2 \int_{-\infty}^{\infty} u^*(x)\, \frac{\mathrm{d}^2}{\mathrm{d}x^2}\, u(x)\, \mathrm{d}x$$

$$= -\hbar^2\, C^2 \int_{-\infty}^{\infty} e^{-x^2/2a^2}\, \frac{\mathrm{d}^2}{\mathrm{d}x^2}\, e^{-x^2/2a^2}\, \mathrm{d}x = -\hbar^2\, C^2 \int_{-\infty}^{\infty} e^{-x^2/2a^2}\, \frac{\mathrm{d}}{\mathrm{d}x} \left[\left(\frac{-x}{a^2} \right) e^{-x^2/2a^2} \right] \mathrm{d}x$$

$$= -\hbar^2\, C^2 \int_{-\infty}^{\infty} e^{-x^2/2a^2} \left[\left(\frac{-1}{a^2} \right) + \left(\frac{x^2}{a^4} \right) \right] e^{-x^2/2a^2}\, \mathrm{d}x$$

$$= \frac{\hbar^2\, C^2}{a^2} \left[\int_{-\infty}^{\infty} e^{-x^2/a^2}\, \mathrm{d}x - \int_{-\infty}^{\infty} \frac{x^2}{a^2}\, e^{-x^2/a^2}\, \mathrm{d}x \right]$$

$$= \frac{\hbar^2}{a^2} \left[C^2 \int_{-\infty}^{\infty} e^{-x^2/a^2}\, \mathrm{d}x - \frac{C^2}{a^2} \int_{-\infty}^{\infty} x^2\, e^{-x^2/a^2}\, \mathrm{d}x \right]$$

$$\text{from normalization} \qquad C^2 \int_{-\infty}^{\infty} e^{-x^2/a^2}\, \mathrm{d}x = 1$$

$$\text{and from part b)} \qquad C^2 \int_{-\infty}^{\infty} x^2\, e^{-x^2/a^2}\, \mathrm{d}x \;=\; \frac{C^2 a^3 \sqrt{\pi}}{2}$$

$$\Rightarrow \quad \langle p^2 \rangle = \frac{\hbar^2}{a^2} \left[1 - \frac{C^2 a \sqrt{\pi}}{2} \right] = \frac{\hbar^2}{a^2} \left[1 - \frac{1}{2} \right] = \frac{\hbar^2}{2a^2}$$

6.3 Show that, for the the Gaussian wave function $u(x) = Ce^{-x^2/2a^2}$, the product $\langle x^2 \rangle \langle p^2 \rangle$ is independent of a and is proportional to $\hbar^2$.

From the previous problem:

$$\langle x^2 \rangle = \frac{a^2}{2} \quad \text{and} \quad \langle p^2 \rangle = \frac{\hbar^2}{2a^2} \quad \Rightarrow \quad \langle x^2 \rangle \langle p^2 \rangle = \frac{\hbar^2}{4}$$

6.12 Consider an electron in an infinite well of width L and in an energy eigenstate characterized by the quantum number n. Calculate $\langle x^2\rangle$ and $\langle p^2\rangle$. Show that the product $\langle x^2\rangle\langle p^2\rangle$ is indenpendent of L. How does the product depend on n?

$$u(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{p_n x}{\hbar}\right)$$

$$\langle x^2\rangle = \int\limits_{-\infty}^{\infty} u^*(x)\,\hat{x}^2\,u(x)\,\mathrm{d}x = \int\limits_{-\infty}^{\infty} u^*(x)\,x^2\,u(x)\,\mathrm{d}x = \frac{2}{L}\int\limits_{0}^{L} x^2\sin^2\left(\frac{p_n x}{\hbar}\right)\mathrm{d}x$$

$$= \frac{2}{L}\int\limits_{0}^{L} x^2\sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left(\frac{2}{L}\right)\left(\frac{L}{n\pi}\right)^3\int\limits_{0}^{n\pi}\theta^2\sin^2\theta\,\mathrm{d}\theta \qquad \text{with} \qquad \theta = \frac{n\pi x}{L}$$

Look this integral up in an integral table

$$\langle x^2\rangle = \frac{2L^2}{(n\pi)^3}\left[\frac{\theta^3}{6} - \left(\frac{\theta^2}{4} - \frac{1}{8}\right)\sin(2\theta) - \frac{\theta\cos(2\theta)}{4}\right]\Bigg|_0^{n\pi} = L^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} u^*(x)\,\hat{p^2}\,u(x)\,\mathrm{d}x = -\hbar^2\int_{-\infty}^{\infty} u^*(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,u(x)\,\mathrm{d}x$$

$$= \left(\frac{2\hbar^2}{L}\right)\left(\frac{p_n}{\hbar}\right)^2\int_0^L \sin^2\left(\frac{p_n x}{\hbar}\right)\,\mathrm{d}x = \left(\frac{2\,p_n^2}{L}\right)\int_0^L \sin^2\left(\frac{p_n x}{\hbar}\right)\,\mathrm{d}x$$

$$= \left(\frac{2\,p_n^2}{L}\right)\left(\frac{\hbar}{p_n}\right)\int_0^L \sin^2\left(\frac{p_n x}{\hbar}\right)\,\mathrm{d}x = \left(\frac{2\,p_n\hbar}{L}\right)\int_0^L \sin^2\left(\frac{p_n x}{\hbar}\right)\,\mathrm{d}x$$

$$= \left(\frac{2\,n\pi\hbar^2}{L^2}\right)\int_0^{n\pi}\sin^2\theta\,\mathrm{d}\theta \qquad \text{with} \qquad \theta = \frac{n\pi x}{L}$$

$$\langle p^2\rangle = \left(\frac{n\pi\hbar}{L}\right)^2$$

$$\langle x^2\rangle\langle p^2\rangle = L^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right]\left(\frac{n\pi\hbar}{L}\right)^2 = \hbar^2\left[\frac{n^2\pi^2}{3} - \frac{1}{2}\right]$$

which is independent of L

6.14 Consider a wave function of the form $\psi(x) = Au_1(x) + Bu_2(x)$ where $u_1(x)$ and $u_2(x)$ are eigenfunctions of the infinite-well potential.

a) Use the requirement that $\psi(x)$ is properly normalized to show that $|A|^2 + |B|^2 = 1$.

$$\text{Using} \quad \int_0^L u_n^*(x) u_m(x)\,\mathrm{d}x = 0 \quad \text{for} \quad n \neq m$$

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,\mathrm{d}x = 1$$

$$1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \left[A^* u_1^*(x) + B^* u_2^*(x)\right]\left[Au_1(x) + Bu_2(x)\right]\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left[A^*Au_1^*(x)u_1(x) + A^*Bu_1^*(x)u_2(x) + B^*Au_2^*(x)u_1(x) + B^*Bu_2^*(x)u_2(x)\right]\mathrm{d}x$$

$$= |A|^2 + |B|^2$$

$$\text{Since} \quad \int_0^L u_n^*(x) u_m(x)\,\mathrm{d}x = 0 \quad \text{for} \quad n \neq m$$

$$\text{and} \quad \int_0^L u_n^*(x) u_m(x)\,\mathrm{d}x = 1 \quad \text{for} \quad n = m$$

b) Calculate the expectation value of the energy $p^2/2m$ for this wave function and show that it is equal to

$$\langle E\rangle = |A|^2 E_1 \, + \, |B|^2 E_2,$$

where E_1 and E_2 are the energy values corresponding to $n = 1$ and $n = 2$, respectively.

$$\langle E\rangle = \frac{\langle p^2\rangle}{2m} = \frac{1}{2m} = -\frac{\hbar^2}{2m}\int\limits_{-\infty}^{\infty} \psi^*(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\psi(x)\,\mathrm{d}x$$

$$= -\frac{\hbar^2}{2m}\int\limits_{-\infty}^{\infty} \left[A^* u_1^*(x) + B^* u_2^*(x)\right]\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\left[A u_1(x) + B u_2(x)\right]\,\mathrm{d}x$$

$$= -\frac{\hbar^2}{2m}\int\limits_{-\infty}^{\infty} \left[A^* \sin\left(\frac{p_1 x}{\hbar}\right) + B^* \sin\left(\frac{p_2 x}{\hbar}\right)\right]\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\left[A \sin\left(\frac{p_1 x}{\hbar}\right) + B \sin\left(\frac{p_2 x}{\hbar}\right)\right]\,\mathrm{d}x$$

$$= \frac{\hbar^2}{2m}\int\limits_{-\infty}^{\infty} \left[A^* \sin\left(\frac{p_1 x}{\hbar}\right) + B^* \sin\left(\frac{p_2 x}{\hbar}\right)\right]\left[\left(\frac{p_1}{\hbar}\right)^2 A \sin\left(\frac{p_1 x}{\hbar}\right) + \left(\frac{p_2}{\hbar}\right)^2 B \sin\left(\frac{p_2 x}{\hbar}\right)\right]\,\mathrm{d}x$$

$$= \frac{\hbar^2}{2m}\int\limits_{-\infty}^{\infty} \left[A^* u_1^*(x) + B^* u_2^*(x)\right]\left[\left(\frac{p_1}{\hbar}\right)^2 A u_1(x) + \left(\frac{p_2}{\hbar}\right)^2 B u_2(x)\right]\,\mathrm{d}x$$

$$= \frac{1}{2m}\int\limits_{-\infty}^{\infty} \left(|A|^2\, p_1^2\, u_1^*(x) u_1(x) + A^* B\, p_2^2\, u_1^*(x) u_2(x) + B^* A\, p_1^2\, u_2^*(x) u_1(x) + |B|^2\, p_2^2\, u_2^*(x) u_2(x)\right)\mathrm{d}x$$

$$= \frac{1}{2m}\left(|A|^2\, p_1^2 + |B|^2\, p_2^2\right) = |A|^2 E_1 \, + \, |B|^2 E_2$$

7.1 Consider the very simple wave packet

$$\psi(x) = A\exp\left[\frac{i(p+\Delta p)x}{\hbar}\right] + A\exp\left[\frac{i(p-\Delta p)x}{\hbar}\right]$$

Show that $\psi(x)$ takes the form of a single wave of momentum p multiplied by a function. What is the function?

$$\psi(x) = A\exp\left[\frac{i(p+\Delta p)x}{\hbar}\right] + A\exp\left[\frac{i(p-\Delta p)x}{\hbar}\right]$$

$$= Ae^{ipx/\hbar}\left(e^{i\Delta px/\hbar} + e^{-i\Delta px/\hbar}\right) = 2Ae^{ipx/\hbar}\cos\left(\frac{\Delta px}{\hbar}\right)$$

The function is $\cos\left(\frac{\Delta px}{\hbar}\right)$

7.3 Consider the time-dependent wave packet

$$\psi(x,t) = A\exp\left[\frac{i(p_1x - E_1t)}{\hbar}\right] + A\exp\left[\frac{i(p_2x - E_2t)}{\hbar}\right]$$

where $p_1 = p + \Delta p, \quad p_2 = p - \Delta p, \quad E_1 = E + \Delta E,$ and $E_2 = E - \Delta E.$

a) Show that $\psi(x,t)$ takes the form of a plane wave times a time-dependent modulating factor.

$$\psi(x,t) = A\exp\left[\frac{i(p_1x - E_1t)}{\hbar}\right] + A\exp\left[\frac{i(p_2x - E_2t)}{\hbar}\right]$$

$$= A\exp\left[\frac{i(px - Et) + i(\Delta px - \Delta Et)}{\hbar}\right] + A\exp\left[\frac{i(px - Et) - i(\Delta px - \Delta Et)}{\hbar}\right]$$

$$= Ae^{i(px-Et)/\hbar}\left(e^{i(\Delta px - \Delta Et)/\hbar} + e^{-i(\Delta px - \Delta Et)/\hbar}\right)$$

$$= 2Ae^{i(px-Et)/\hbar}\cos\left(\frac{\Delta px - \Delta Et}{\hbar}\right)$$

b) Show that the modulating factor has a time dependence that can be interpreted as the propagation of an "envelope" moving with the speed $v = \Delta E/\Delta p$.

The modulating factor is the form of wave with wave number, $k = \Delta p/\hbar$, and angular frequency, $\omega = \Delta E/\hbar$.

$$\cos\left(\frac{\Delta p x - \Delta E t}{\hbar}\right) = \cos\left(\frac{kx - \omega t}{\hbar}\right)$$

$\Rightarrow$ the velocity of the modulation is:

$$v = \frac{\omega}{k} = \frac{\Delta E}{\Delta p}$$

7.4 Consider the definition of the group velocity

$$v_g = \frac{\mathrm{d}E}{\mathrm{d}p}$$

Show that for the propagation of light, the group velocity is the speed of light.

$$\boxed{\text{for photon} \qquad E = pc \qquad \Rightarrow \qquad \frac{\mathrm{d}E}{\mathrm{d}p} = c}$$

7.8 Suppose one wishes to use an electron beam to resolve a molecule whose size is 0.8 nm. What is the kinetic energy of electrons that have the requisite wavelength?

$$\boxed{\begin{gathered} \Delta x \sim \frac{h}{\Delta p} \sim \frac{h}{p} \qquad \Rightarrow \qquad p \sim \frac{h}{\Delta x} \\ E = \frac{p^2}{2m} = \frac{h^2}{2m(\Delta x)^2} = \frac{(2\pi\hbar c)^2}{2mc^2(\Delta x)^2} \\ = \frac{(4\pi^2)(200\ \text{MeV}\cdot\text{fm})^2}{(2)(0.51\ \text{MeV})(8\times10^{-10}\ \text{m})^2} = 2.4\ \text{eV} \end{gathered}}$$

7.14 The binding energy of an electron in a crystal lattice is 1.2×10^{-4} eV. Use the uncertainty relation to estimate the size of the spread of the electron wave function. How many lattice sites will that include, assuming that the spread is spherically symmetric and that the spacing between the ions is 0.12 nm?

$$E = \frac{p^2}{2m} \qquad \Rightarrow \qquad p = \sqrt{2mE}$$

$$\Delta x \Delta p \approx \frac{\hbar}{2} \qquad \qquad \Delta p \approx p$$

$$\Rightarrow \qquad \Delta x \approx \frac{\hbar}{2\sqrt{2mE}} = \frac{\hbar c}{2\sqrt{2mc^2 E}} = \frac{(200\ \text{MeV}\cdot\text{fm})}{2\sqrt{(2)(0.51\ \text{MeV})(1.2 \times 10^{-4}\ \text{eV})}}$$

$$= \frac{(2.0 \times 10^{-7}\ \text{eV}\cdot\text{m})}{2\sqrt{(2)(5.1 \times 10^{5}\ \text{eV})(1.2 \times 10^{-4}\ \text{eV})}} = 9.0 \times 10^{-9}\ \text{m}$$

$$\text{radius} = \frac{\Delta x}{2} = 4.5 \times 10^{-9}\ \text{m}$$

$$= \frac{4.5 \times 10^{-9}\ \text{m}}{1.2 \times 10^{-10}\ \text{m}}\ \text{atomic spacings} = 37\ \text{atomic spacings}$$

$$\text{spherical volume contains} \qquad \frac{4}{3}\pi(37)^3 = 2.1 \times 10^{5}\ \text{lattice points}$$

7.17 An atomic state has a mean life of 2.6×10^{-10} s. What is the uncertainty in the energy value of the state?

$$\boxed{\begin{gathered}\Delta E \Delta t \approx \frac{\hbar}{2} \qquad \Rightarrow \qquad \Delta E \approx \frac{\hbar}{2\tau} = \frac{\hbar c}{2\tau c} \\[2em] = \frac{200 \text{ MeV}\cdot\text{fm}}{(2)(2.6 \times 10^{-10} \text{ s})(3.0 \times 10^{8} \text{ m/s})} = \frac{2.0 \times 10^{-7} \text{ eV}\cdot\text{m}}{(2)(2.6 \times 10^{-10} \text{ s})(3.0 \times 10^{8} \text{ m/s})} \\[2em] = 1.3 \times 10^{-6} \text{ eV}\end{gathered}}$$