1. Question 1

a) In the center of momentum frame, \( \vec{p}_A = -\vec{p}_B \), where \( A \) and \( B \) arbitrarily label the two incoming protons. Furthermore, the momenta of the incoming protons are equal and opposite, so \( p \equiv |\vec{p}_A| = |\vec{p}_B| \), and consequently \( E \equiv E_A = E_B \), therefore
\[
\vec{p}_A = (p, 0, 0, iE/c),
\]
and
\[
\vec{p}_B = (-p, 0, 0, iE/c).
\]
From the definition of \( s \), we have
\[
s = -(\vec{p}_A + \vec{p}_B)^2/c^2 = 4E^2/c^4
\]
so solving for \( E \) gives
\[
E = \frac{1}{2} \sqrt{sc^2}.
\]
\( \sqrt{sc^2} = 13 \text{ TeV} \), and thus each proton has an energy of \( \left[ 6.5 \text{ TeV} \right] \).

The gamma factor for a particle is
\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}},
\]
and solving for velocity we have
\[
\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}.
\]
When \( \gamma \gg 1 \), we can Taylor expand the above expression,
\[
\frac{v}{c} \approx 1 - \frac{1}{2 \gamma^2}.
\]
Equivalently, the difference between speed of light travel and \( v/c \) is
\[
1 - \frac{v}{c} \approx \frac{1}{2 \gamma^2}.
\]
Since
\[
\gamma = \frac{E}{mc^2} \approx \frac{6.5 \times 10^6 \text{ MeV}}{938 \text{ MeV}} \approx 6.93 \times 10^3,
\]
we can use the equation derived above to get the corresponding velocity:
\[
1 - \frac{v}{c} \approx \frac{1}{2 \gamma^2} \approx 1.04 \times 10^{-8}.
\]
So the velocity of the proton is 

\[ v \approx (1 - 1.04 \times 10^{-8}) \, c, \]

or equivalently, 3 m/s shy of the velocity of light.

b) The velocities of the protons are equal and opposite, so using the relativistic velocity addition equation gives 

\[ u' = \frac{2 \, u}{1 + \frac{u^2}{c^2}}. \]

Rewriting this expression with \( u = (1 - x)c \), we get 

\[ u' = \frac{2(1 - x)c}{1 + (1 - x)^2} = \frac{2 - 2x}{2 - 2x + x^2}c = \frac{1}{1 + \frac{x^2}{2 - 2x}}c. \]

Taylor expanding about \( x^2/(2 - 2x) \) gives 

\[ u' \approx \left(1 - \frac{x^2}{2 - 2x}\right)c, \]

and since \( 2x \ll 1 \), 

\[ u' \approx \left(1 - \frac{x^2}{2}\right)c. \]

Plugging in \( x = 1.04 \times 10^{-8} \) gives a velocity for one proton in the rest frame of the other as 

\[ u' \approx (1 - 5.4 \times 10^{-17})c, \]

or equivalently, 16 nm/s shy of the velocity of light. We can use the earlier derived expression to get \( \gamma \): 

\[ 1 - \frac{v}{c} \approx \frac{1}{2 \, \gamma^2}, \]

so 

\[ \frac{x^2}{2} \approx \frac{1}{2 \, \gamma^2} \]

and thus \( \gamma \approx 1/x \approx 9.6 \times 10^7 \). This corresponds to a proton energy of 

\[ E = \gamma mc^2 = (9.6 \times 10^7)(938 \text{ MeV}) \approx 9.0 \times 10^9 \text{ MeV}. \]

2. Question 2

a) For an arbitrary \( n \), the number of choices for each quark or antiquark is \( n \). The choice of quark has no bearing on the choice of the antiquark, so to form a \( q\bar{q} \) bound state, there are \( \left[ \binom{n}{2} \right] \) choices. Thus there are 1, 4, 9, 16, 25, and 36 possibilities for \( n = 1, 2, 3, 4, 5, \) and 6, respectively.

b) For an arbitrary \( n \), the number of choices for each quark in a baryon \((qqq)\) bound state is not \( n^3 \) because there are equivalent permutations. The total can be broken down as follows. If all three quarks are the same, then there are \( n \) possibilities. If two quarks are
the same, then there are \( n(n - 1) \) possibilities. If no two quarks are the same, then there are \( n(n - 1)(n - 2)/6 \) possibilities. The reason to divide by six is to cover the six equivalent permutations, e.g., \((u,d,s)\), \((u,s,d)\), \((d,u,s)\), \((d,s,u)\), \((s,u,d)\), \((s,d,u)\). So the total is

\[
\begin{align*}
&= n + n(n - 1) + n(n - 1)(n - 2)/6 \\
&= n^2 + n(n - 1)(n - 2)/6 \\
&= [6n + (n - 1)(n - 2)] \frac{n}{6} \\
&= [6n + n^2 - 3n + 2] \frac{n}{6} \\
&= [n^2 + 6n + 2] \frac{n}{6} \\
&= \left[ n(n + 1)(n + 2)/6 \right]
\end{align*}
\]

Thus there are 1, 4, 10, 20, 35, and 56 combinations for \( n = 1, 2, 3, 4, 5, \) and 6 choices, respectively. Including the possibility of anti-baryons (\( q\bar{q}q \) bound states), would require multiplying by 2.

3. Question 3

a) violates charge conservation
b) violates energy/momentum conservation
c) possible
d) possible
e) violates baryon number
f) violates baryon and lepton (\( L_e \)) number
g) possible
h) possible, strangeness violating
i) possible, strangeness violating
j) possible, (doubly!) strangeness violating
k) possible
l) violates energy/momentum conservation
m) possible, strangeness violating
n) violates lepton (\( L_\mu \)) number
o) possible
p) possible (note: this is not strangeness violating)
q) possible
r) violates charge conservation
s) violates baryon number
t) possible, strangeness violating
u) possible (note: this is not strangeness violating)
v) possible, strangeness violating
w) possible
4. QUESTION 4

Using four vectors is not strictly necessary here since we are not evaluating the system in multiple frames. In the lab frame, momentum conservation (in the longitudinal direction) requires that

\[ p = k \cos \phi + p_e \cos \theta, \]

where \( k \) is the momentum of the incident particle after the collision, \( \phi \) is the scattered angle with respect to the initial trajectory, and \( p_e \) is the magnitude of the momentum of the electron. Momentum conservation (in the transverse direction) also requires that

\[ k \sin \phi = p_e \sin \theta. \]

Energy conservation requires that

\[ \sqrt{p^2 c^2 + M^2 c^4 + m_e c^2} = \sqrt{k^2 c^2 + M^2 c^4 + E}. \]

We can solve for \( k^2 \) and remove the reference to \( \phi \) by re-arranging and squaring the first momentum conservation equation,

\[ (p - p_e \cos \theta)^2 = k^2 \cos^2 \phi, \]

and adding to it the square of the second momentum conservation equation

\[ (p - p_e \cos \theta)^2 + p_e^2 \sin^2 \theta = k^2 \cos^2 \phi + k^2 \sin^2 \phi, \]

and so

\[ k^2 = p^2 + p_e^2 - 2pp_e \cos \theta. \]

We can plug this result into the energy conservation equation to get

\[ \sqrt{p^2 c^2 + M^2 c^4} = \sqrt{p_e^2 c^2 - 2pp_e c^2 \cos \theta + M^2 c^4 + E}. \]

Moving \( E \) from the RHS to the LHS and squaring gives

\[ p^2 c^2 + M^2 c^4 + (E - m_e c^2)^2 - 2(E - m_e c^2) \sqrt{p^2 c^2 + M^2 c^4} = p^2 c^2 + p_e^2 c^2 - 2pp_e c^2 \cos \theta + M^2 c^4. \]

The \( p^2 c^2 \) and \( M^2 c^4 \) terms on both sides cancel, and we can rearrange terms to give

\[ \sqrt{p^2 c^2 + M^2 c^4} = \frac{p_e^2 c^2 - 2pp_e c^2 \cos \theta - (E - m_e c^2)^2}{-2(E - m_e c^2)}. \]

Noting that \( E^2 = p_e^2 c^2 + m_e c^4 \), we get

\[ \sqrt{p^2 c^2 + M^2 c^4} = \frac{E^2 - m_e^2 c^4 - 2pc \sqrt{E^2 - m_e^2 c^4} \cos \theta - E^2 - m_e^2 c^4 + 2E m_e c^2}{-2(E - m_e c^2)} \]

\[ = \frac{pc \sqrt{E^2 - m_e^2 c^4} \cos \theta - E m_e c^2 + m_e^2 c^4}{E - m_e c^2} \]

\[ = pc \cos \theta \sqrt{\frac{E + m_e^2 c^2}{E - m_e^2 c^2} - m_e^2}. \]
Solving for $M$ gives

$$Mc^2 = \sqrt{\left(p_c \cos \theta \sqrt{\frac{E + m_e^2c^2}{E - m_e^2c^2} - m_e^2c^2}\right)^2 - p^2c^2},$$

which we can alternatively write as

$$M = \frac{p}{c} \left[\left(\cos \theta \sqrt{\frac{E + m_e^2c^2}{E - m_e^2c^2} - \frac{m_e^2c^2}{pc}}\right)^2 - 1\right]^{\frac{1}{2}}.$$

Since we are interested in the case where $pc \gg m_e^2c^2$, this could be numerically approximated as

$$M \approx \frac{p}{c} \left[\frac{E + m_e^2c^2}{E - m_e^2c^2} \cos^2 \theta - 1\right]^{\frac{1}{2}}.$$

Plugging in $p = 51$ MeV/c, $E = \sqrt{0.11^2 + 0.511^2} \approx 0.523$ MeV, and $\cos \theta \approx 0.940$ to the above equation gives $M \approx 460$ MeV/c$^2$. This is approximately the mass of the charged kaon, $K^\pm$.

5. Question 5

From the uncertainty principle

$$\Delta p_x \geq \frac{h}{2\Delta x},$$

We constrain the electron to have $\Delta x = 1$ fm.

$$\Delta p_x \geq \frac{6.582 \times 10^{-16} \text{eV} \cdot \text{s}}{2 \text{fm}} = \frac{(6.582 \times 10^{-16} \text{eV} / \text{c} \cdot \text{s})(3 \times 10^8 \text{m/s})}{2 \times 10^{-15} \text{m}} \approx 98.7 \text{MeV/c}.$$

Considerations of the other two spatial dimensions would yield an additional multiplicative factor of $\sqrt{3}$, but this is hardly important given the outcome. The corresponding energy is

$$E = \sqrt{p^2c^2 + m_e^2c^4} \approx 98.7 \text{MeV},$$

and the corresponding kinetic energy is

$$K = E - m_e^2c^2 = 98.2 \text{MeV},$$

where $m_e = 0.511 \text{MeV/c}^2$. This is several orders of magnitude greater than 6 keV, which is the typical kinetic energy of an electron from the beta decay of tritium. The electron’s maximum kinetic energy in tritium decay is 17 keV, so in any case, it seems unlikely that the electron is contained within the nucleus.