1. **Question 1**

The Coulomb potential between a nucleus of charge $Z$ and an electron is

$$V(r) = -\frac{Ze^2}{r},$$

so

$$E_n(Z) = -\left(\frac{m_e}{2\hbar^2} \left( \frac{Ze^2}{4\pi\epsilon_0} \right)^2 \right) \frac{1}{n^2} = \frac{Z^2 E_n}{Z^2}.$$  

Similarly,

$$a(Z) = \frac{4\pi\epsilon_0\hbar^2}{m_eZe^2} = \frac{a}{Z}.$$  

The reduced mass of a proton-electron system is

$$\mu = \frac{m_em_p}{m_e + m_p} = \frac{m_e}{1 + m_e/m_p} \approx m_e \left(1 - \frac{m_e}{m_p}\right).$$

Since

$$\frac{m_e}{m_p} \approx \frac{0.511 \text{ MeV}/c^2}{938 \text{ MeV}/c^2} \approx 5.4 \times 10^{-4},$$

the mass of the nucleus has at most a $\sim 5 \times 10^{-4}$ effect on the predicted energies.

2. **Question 2**

   a) $0! \ Y^\ell_\ell$ is already at the highest value of $m$ possible.

   b) Let $L_zY^\ell_\ell = \Theta(\theta)\Phi(\phi)$ so that $L_zY^\ell_\ell = \hbar \ell Y^\ell_\ell$ can be written as

   $$\frac{\hbar}{i} \Theta \frac{\partial \Phi}{\partial \theta} = \hbar \ell \Theta \Phi.$$  

   Dividing out by $\hbar \Theta / i$ gives

   $$\frac{\partial \Phi}{\partial \phi} = i\ell \Phi,$$

   which is solved by $\Phi(\phi) = e^{i\ell \phi}$, where we absorbing the normalization constant into the function $\Theta(\theta)$. Similarly, the expression $L_+Y^\ell_\ell = 0$ can be written as the following differential equation:

   $$\hbar e^{i\phi} \left( \Phi \frac{\partial \Theta}{\partial \theta} + i\Theta \cot \theta \frac{\partial \Phi}{\partial \phi} \right) = 0.$$
Plugging in our solution for $\Phi$ gives

$$\hbar e^{i\phi} \Phi \left( \frac{\partial \Theta}{\partial \theta} - \ell \Theta \cot \theta \right) = 0,$$

which only holds generally for $\Phi$ when

$$\frac{d\Theta}{d\theta} = \ell \cot \Theta,$$

which we can re-write as

$$\frac{d\Theta}{\Theta} = \ell \cot d\theta.$$

Integrating both sides gives

$$\int \frac{d\Theta}{\Theta} = \int \ell \cot d\theta$$

so that

$$\ln \Theta = \ell \ln \sin \theta + C,$$

and hence

$$\Theta = A \sin^\ell \theta.$$

This means that

$$Y_\ell^\ell(\theta, \phi) = A \left( \sin \theta e^{i\phi} \right)^\ell,$$

where $A$ is the normalization constant.

c) The normalization condition is

$$A^2 \int_0^\pi \int_0^{2\pi} \sin^{2\ell} \theta \sin \theta d\theta d\phi = 1.$$

Integrating out $\phi$ gives

$$2\pi A^2 \int_0^\pi \sin^{2\ell+1} \theta d\theta = 1.$$

We computed this integral before on question 2c of HW7:

$$\int_0^\pi (\sin \theta)^{2\ell+1} d\theta = 2 \frac{(2\ell)!}{(2\ell + 1)!},$$

so

$$Y_\ell^\ell(\theta, \phi) = \frac{1}{\ell!} \sqrt{\frac{(2\ell + 1)!}{4\pi}} \left( \frac{1}{2} e^{i\phi} \sin \theta \right)^\ell,$$

which is exactly what we had on question 2b of HW7 up to an arbitrary normalization of $(-1)^\ell$.

d) Applying the raising operator to $Y_1^2$ gives

$$\hbar e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) \left( -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i\phi} \right) = 2\hbar Y_2^2.$$
where we have used the fact that the normalization term $\hbar\sqrt{\ell(\ell+1) - m(m+1)} = 2\hbar$. Simplifying this expression gives

$$Y_2^2(\theta, \phi) = -\sqrt{\frac{15}{32\pi}} e^{i\phi} \left( e^{i\phi}(\cos^2\theta - \sin^2\theta) - e^{i\phi}\cos^2\theta \right)$$

$$= \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{2i\phi}.$$

3. **Question 3**

a) The kinetic energy of the two particles is $2\cdot \frac{1}{2}mv^2$, and the total angular momentum $|L| = 2\cdot \frac{1}{2}mv$. In terms of the angular momentum, the energy of the system is

$$H = \frac{L^2}{ma^2}.$$ 

The eigenvalues of $L^2$ are $\hbar^2\ell(\ell+1)$, and since $ma^2$ is a constant, we have the eigenvalues of $H$ are the energies

$$E_n = n(n+1)\frac{\hbar^2}{ma^2}$$

for $n = 0, 1, 2, \ldots$, where we have labeled the quantum numbers $n$ instead of $\ell$.

b) The normalized eigenvalues of the wavefunction are simply $\psi_{nm}(\theta, \phi) = Y_n^m(\theta, \phi)$, that is the spherical harmonics. The degeneracy is therefore $d(n) = 2n + 1$.

4. **Question 4**

a) The 3D harmonic oscillator in Cartesian coordinates gives the time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m} \left[ \frac{\partial^2 X}{\partial x^2} YZ + X \frac{\partial^2 Y}{\partial y^2} Z + XY \frac{\partial^2 Z}{\partial z^2} \right] + \frac{1}{2}m\omega^2 XYZ \left[ x^2 + y^2 + z^2 \right] = E \cdot XYZ$$

where we have substituted

$$\psi(x, y, z) = X(x)Y(y)Z(z).$$

Dividing out by $XYZ$ gives

$$\left\{ -\frac{\hbar^2}{2m} \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \right\} + \left\{ -\frac{\hbar^2}{2m} \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{2}m\omega^2 y^2 \right\} + \left\{ -\frac{\hbar^2}{2m} \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} + \frac{1}{2}m\omega^2 z^2 \right\} = E,$$

where the first term depends only on $x$, the second only on $y$, and the third only on $z$. So they differ with respect to each other only by the constants $E_x$, $E_y$, and $E_z$, such that
\[ E_x + E_y + E_z = E. \] The energy for the 1D harmonic oscillator is \( E_n = \left( \frac{1}{2} + n \right) \hbar \omega, \) so by analogy,

\[ E = \hbar \omega \left[ \left( \frac{1}{2} + n_x \right) + \left( \frac{1}{2} + n_y \right) + \left( \frac{1}{2} + n_z \right) \right] = \left( n + \frac{3}{2} \right) \hbar \omega, \]

where \( n = n_x + n_y + n_z, \) and \( n = 0, 1, 2, \ldots \)

(b) The degeneracy is determined by how many ways three non-negative integers can sum to \( n. \) Two non-negative integers can sum up to \( n + 1 \) ways: 0 + \( n, \) 1 + (\( n - 1 \)), 2 + (\( n - 2 \)), \ldots, (\( n - 1 \)) + 1, \( n + 0. \) So the number of ways that two numbers can sum up to \( n - n_x \) is \( n - n_x + 1. \) If \( n_x = n, \) then there is 1 way, if \( n_x = n - 1 \) there are 2 ways, if \( n_x = n - 2 \) there are 3 ways, etc. if \( n_x = 0, \) there are \( n + 1 \) ways. So there are a total of \( 1+2+3+\cdots+n+1 \) ways to make three non-negative integers sum up to \( n. \) Therefore

\[ d(n) = \frac{(n + 1)(n + 2)}{2}. \]

5. Question 5

Plugging into the given equation for the classical radius, we get

\[ r_c = \frac{(1.6 \times 10^{-19})^2}{4\pi(8.85 \times 10^{-12})(9.11 \times 10^{-31})(3.0 \times 10^8)^2} = 2.81 \times 10^{-15} \text{ m}. \]

We also know that

\[ L = \frac{1}{2} \hbar = I \omega = \left( \frac{2}{5} mr^2 \right) \left( \frac{v}{r} \right) = \frac{2}{5} mrv, \]

where we have used the moment of inertia for a solid sphere. Thus,

\[ v = \frac{5\hbar}{4mr} = \frac{(5)(1.055 \times 10^{-34})}{(4)(9.11 \times 10^{-31})(2.81 \times 10^{-15})} = 5.2 \times 10^{10} \text{ m/s}. \]

Since the speed of light is only \( 3 \times 10^8 \text{ m/s}, \) the point on the “equator” would be traveling more than 100 times faster than light, so this model does not make sense.